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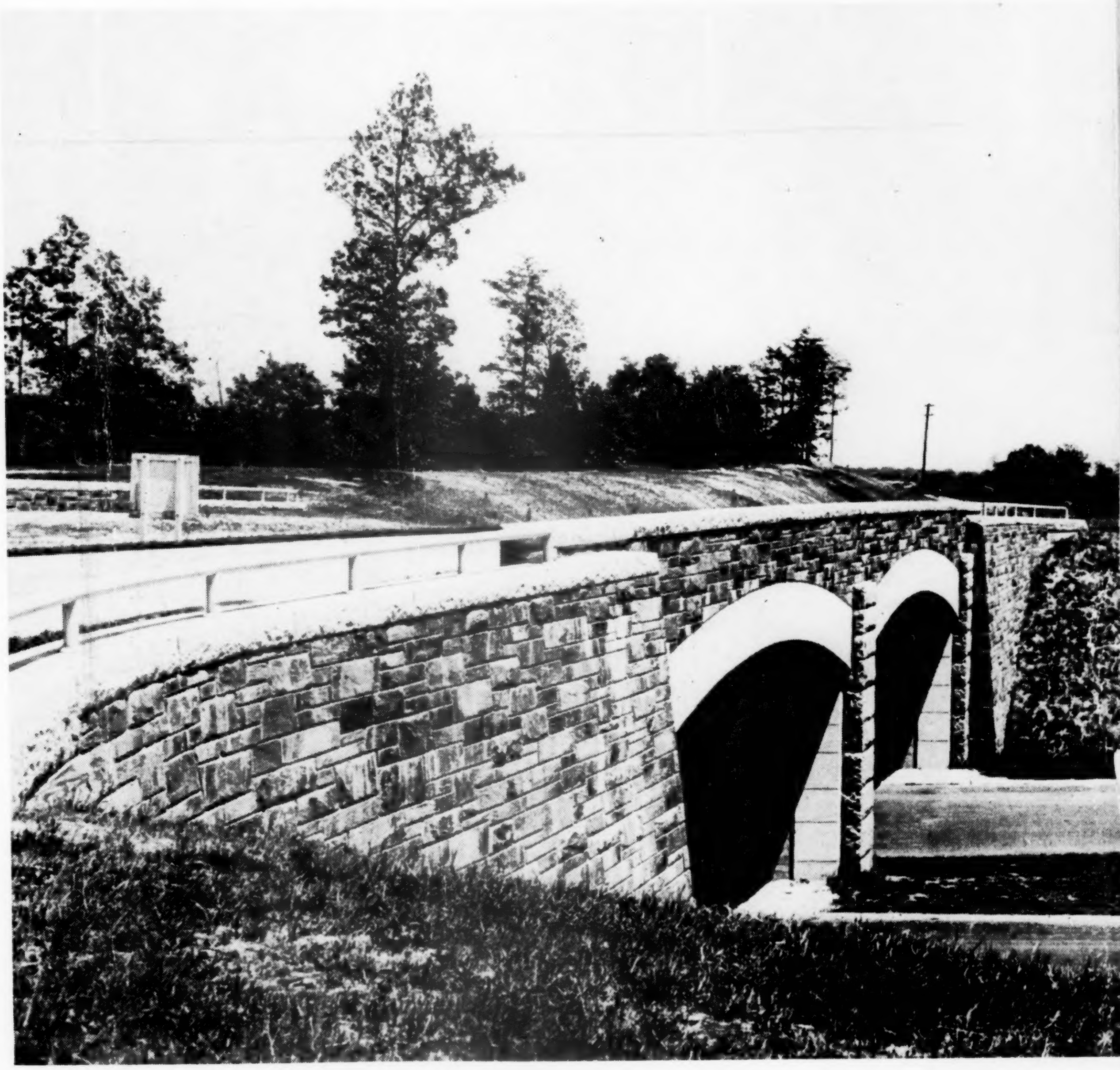

Grade separation on access road to Andrews Air Field, Maryland

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BUREAU OF PUBLIC ROADS

U. S. DEPARTMENT OF COMMERCE

E. A. STROMBERG, Editor

Moment Distribution Analysis of Two-Span Arched Frames With Elastic Pier

By THOMAS P. REVELISE, [1] **Highway Bridge Engineer, Bureau of Public Roads**

During recent years multiple arches in continuous series have assumed considerable importance in the structural engineering field, and several methods of analysis of such structures by moment and thrust distribution have been developed or proposed. This article presents an adaptation of the method of moment distribution to the analysis of two-span arched frames with elastic center pier and with either fixed or hinged footings. The method is applicable to any two-span continuous arch, but is arranged particularly for convenience in the analysis of arched frames because of the increasingly frequent use of this type of structure for grade separations of divided highways.

Part I of the article is devoted to the necessary mathematical development for a structure with hinged footings, a discussion of procedure, and an actual sample analysis of an unsymmetrical two-span frame with hinged footings. In part II expressions for a structure with fixed footings are developed, followed by a discussion of procedure and a sample analysis of the same structure as that used in part I, but with footings fixed.

The use of forms for tabulating computations makes most of the analysis procedure a mechanical operation by which results can be obtained rapidly and accurately by designers of limited experience.

INTRODUCTION

DURING recent years multiple arches in continuous series have assumed considerable importance in the structural engineering field, and several methods of analysis of such structures by moment and thrust distribution have been developed or proposed.[2]

This paper presents an adaptation of the method of moment distribution to the analysis of two-span arched frames with elastic center pier and with either fixed or hinged footings. The method is applicable to any two-span continuous arch, but is arranged for convenience in the analysis of arched frames due to the importance of this type in divided highway overcrossings. Detailed analyses of an unsymmetrical structure with hinged footings, and of the same structure with fixed footings, illustrate the procedure and facilitate its use in the design office with a minimum of preliminary study of the text or reference to other sources.

Criterion for Arch Analysis

It is first desirable to establish a criterion of deck curvature in order to differentiate arched frames requiring an arch analysis from those that may be analyzed as straight frames with empirical corrections for the effect of arch action. Investigations of this subject based on the application of both methods to a number of typical structures show that when the rise of the deck neutral axis line exceeds approximately one twenty-fifth of the design span, commonly used empirical formulas are not valid. An example of the sensitivity of frames to deck arching is the case of a single-span frame subjected to balanced earth pressure. Under this loading condition a straight frame develops negative moment at the haunch, while a frame identical in every respect except for a deck curvature exceeding the span-rise ratio of 1 to 25 develops a positive moment at the haunch.

[1] Acknowledgment is made to Dudley P. Babcock and T. P. Weston Jr., Highway Bridge Engineers, for checking the computations and for many helpful suggestions and criticisms.

[2] (1) *Continuous Frames of Reinforced Concrete*, by Hardy Cross and Newlin Morgan; John Wiley & Sons, 1932. (2) Discussion by Donald E. Larson of the paper *Analysis of Continuous Frames by Distributing Fixed-End Moment*, by Hardy Cross; p. 127, Transactions of the American Society of Civil Engineers, vol. 96, 1932. (3) *Analysis of Multiple Arches*, by Alexander Hrennikoff; p. 388, Transactions of the American Society of Civil Engineers, vol. 101, 1936.

If a structure is sufficiently arched to develop fairly pronounced arch action, failure to investigate it as an arch may result in error as to the character of the moments as well as to their magnitude. A ratio of design rise to design span of 1 to 25 is therefore recommended as the criterion that should govern the decision whether or not analysis as a true arch is necessary.

Hinged or Fixed Footings

Most bridge frames are founded on material of yielding character and are designed on the assumption of hinged conditions at the footings. The footings may be constructed integrally with the pier and abutment stems, or separated by some device such as lead plates to reduce the degree of fixity at the base.

Occasionally the structure is founded on rock. Full fixity at the footings is assumed in the design in this case since the bases of the pier and abutment stems are usually imbedded in the rock and the excavations made for that purpose are filled with concrete. It is recognized that ideal conditions of restraint are practically unattainable and that the actual condition for most structures is intermediate between hinged and fully fixed. The usual practice, nevertheless, is to base the design on either an ideal hinged condition or an ideal fixed condition, giving due consideration to the character of the foundation and type of footing to be constructed.

Design constants and forms for tabulating computations are developed in this paper for both hinged and fixed footings. In general, the variation in the two procedures is analagous to that which is encountered in the application of ordinary moment distribution to straight-framed structures having hinged and fixed members.

Part I of the paper is devoted to the necessary mathematical development for a hinged condition, a discussion of procedure, and an actual sample analysis of an unsymmetrical two-span frame with hinged footings. In Part II expressions for a fixed condition are developed, followed by a discussion of procedure and a sample analysis of the same structure used in Part I, but with footings fixed.

Steps in the Analysis

In deriving the design constants, the frame leg and contiguous arched deck are treated as a structural unit. A load of unity is placed at 10 points on each arch, and fixed-end moments and thrusts are computed at the juncture of the deck members and pier. The fixed-end moments and thrusts are then distributed at this joint until the desired convergence is reached.

The first step of the analysis, computation of fixed-end moments and thrusts for various positions of a unit load, constitutes solution of a single-span unsymmetrical arch, fixed at the connection with the pier and either hinged or fixed at the footing. Formulas for this computation are derived from the basic elastic equations of rotation and displacement. The resultant expressions are adapted to a form for tabulating computations in which computed values of moment, M, vertical reaction, V, and horizontal thrust, H, are obtained directly at the points of fixity for 10 positions of a unit load on each arch. No sketching of influence lines is necessary. By using unit values in the distribution procedure, only two distributions are required for symmetrical structures, and four if the structure is unsymmetrical. The necessary joint constants are evaluated from expressions derived in the computation for fixed-end moments and thrusts. It is recommended that the computations be made on a calculator, and with a degree of accuracy not less than that indicated in the sample analyses.

After the indeterminate moments and reactions are obtained, further design data may be derived in the same manner as for statically determinate structures. This portion of the work is subject to considerable variation and is omitted in the sample analyses.

Tabulating Forms Used

The use of forms for tabulating computations renders most of the procedure mechanical in nature. Experience with similar forms in arch and arched-frame analysis shows that results can be obtained rapidly and accurately by designers of limited experience. The method is thus applicable directly, without reference to the mathematical derivations.

The analysis of two-span arches and arched frames with elastic pier is especially well adapted to the type of procedure illustrated in the sample analyses. The various operations are, in general, analogous to the procedure of ordinary moment distribution, and the convergence of values is rapid. In view of the comparatively limited variation in the geometric characteristics of this type of structure, it is doubtful that actual cases will occur in which the conversion of the distribution cycles is retarded to an objectionable degree.

Occasionally special architectural treatment of the structure, such as the addition of stone facing, results in a pier of sufficient mass to virtually break flexural continuity at the joint. In such cases it is logical to assume fixity at the joint, and design each arch independently.

Somewhat greater refinement in the computed values could be obtained by placing the unit loads between, instead of at the elastic centers of gravity. It is believed, however, that the accuracy of the method used is not inconsistent with the limitations imposed by uncertainties in basic assumptions and in construction.

Following part II of the paper there appears a series of influence lines (figs. 13 and 14, p. 84), showing the comparative effect of hinged and fixed footings at some of the critical points.

Part I.—ANALYSIS OF STRUCTURES WITH HINGED FOOTINGS

Required Design Constants

Three structural elements are considered in the analysis, as illustrated in figure 1: Unsymmetrical arch AB, unsymmetrical arch BC, and elastic pier BD. The required design constants are defined as follows:

Fixed-end moment.—The moment at B due to a load P or P', if B were completely fixed.

Fixed-end thrust.—The thrust at A or C due to a load P or P', if B were completely fixed.

Moment stiffness of arch rib.—The moment at B necessary to produce a rotation of unity, without horizontal or vertical displacement at B.

Induced thrust of arch rib.—The thrust at A or C induced by a moment of unity at B, without horizontal or vertical displacement at B.

Thrust stiffness of arch rib.—The thrust at B necessary to produce a horizontal displacement of unity at B without vertical displacement or rotation at B.

Induced moment of arch rib.—The moment at B induced by a horizontal thrust of unity at A or C, without vertical displacement or rotation at B.

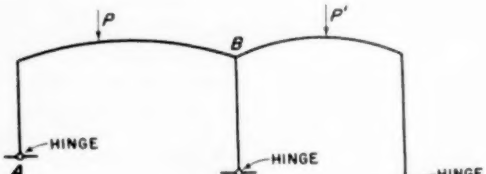


Figure 1.—Design sketch of frame.

Development of Fixed-End Moment and Thrust

In figure 2 the arch rib from the hinge at the footing A to the juncture with the elastic pier and adjacent arch rib at B is treated as a single structural member, fixed at B. The term "arch rib" is used in referring to these members because of the common practice of basing barrel-arch analyses on an element 1 foot wide. This practice is followed in the sample analysis. A concentrated load P is placed on the arch rib, inducing a fixed-end moment at B and equal fixed-end thrusts at B and A. Both the point at which the load P is applied, and the horizontal distance from the neutral axis of the frame leg to that point, are represented as a. In usage, it will be obvious whether a represents the point or the distance.

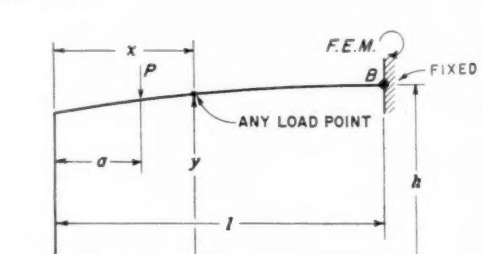


Figure 2.—Sketch for use in deriving fixed-end moment and thrust.

The moment from A to $a = Vx - Hy$.

The moment from a to $B = Vx - Hy - P(x-a)$.

Letting P equal unity, the moment from a to $B = Vx - Hy - (x-a)$.

From the condition that point A is not displaced horizontally and from the condition that point A is not displaced vertically the following two elastic equations may be written:

$$\sum_A^B \frac{Mxds}{EI} \text{ (vertical displacement)} = 0 \quad (1)$$

$$\sum_A^B \frac{Myds}{EI} \text{ (horizontal displacement)} = 0 \quad (2)$$

The modulus of elasticity, E, is constant for the entire structure and may be eliminated from the basic equations in evaluating external reactions due to flexure. In addition, it will simplify the expressions somewhat to use a single symbol for values of ds/I. Accordingly, ds/I is represented by the symbol Δ.

Making these modifications, equations (*1*) and (*2*) may be restated as follows:

$$\sum_A^B Mx\Delta = 0 \quad (3)$$

$$\sum_A^B My\Delta = 0 \quad (4)$$

Inserting the general expression for moment into equations (*3*) and (*4*), the following are obtained:

$$V\sum_A^B x^2\Delta - H\sum_A^B xy\Delta - \sum_a^B (x-a)x\Delta = 0 \quad (5)$$

$$V\sum_A^B xy\Delta - H\sum_A^B y^2\Delta - \sum_a^B (x-a)y\Delta = 0 \quad (6)$$

Solving equations (*5*) and (*6*) for H and V:

$$H = \frac{\sum_a^B (x-a)x\Delta \sum_A^B xy\Delta - \sum_a^B (x-a)y\Delta \sum_A^B x^2\Delta}{\sum_A^B x^2\Delta \sum_A^B y^2\Delta - \left(\sum_A^B xy\Delta\right)^2} \quad (7)$$

$$V = \frac{\sum_a^B (x-a)x\Delta \sum_A^B y^2\Delta - \sum_a^B (x-a)y\Delta \sum_A^B xy\Delta}{\sum_A^B x^2\Delta \sum_A^B y^2\Delta - \left(\sum_A^B xy\Delta\right)^2} \quad (8)$$

When x is less than a, the value of the term $(x-a)$ is zero in equations (*5*), (*6*), (*7*), and (*8*).

The work entailed in evaluating H and V is substantially reduced by a modification of equations (*7*) and (*8*). The assumption is made that the $x\Delta$ and $y\Delta$ values represent actual loads, concentrated at the midpoints of equal dx divisions. Making this assumption, the expressions

$$\sum_a^B (x-a)x\Delta \text{ and } \sum_a^B (x-a)y\Delta$$

in equations (*7*) and (*8*) then represent the cantilever moments about point a of the $x\Delta$ loads and $y\Delta$ loads between a and B. These cantilever moments about point a may in turn be expressed as the areas of the $x\Delta$ and $y\Delta$ shear diagrams between a and B. The ordinate of the $x\Delta$ shear diagram at any load point between a and B is:

$$\sum_{a+1}^B x\Delta$$

In this expression a is any load point between the load and B. The summation is taken from $a+1$ (a plus one load point) because the $x\Delta$ concentration at point a theoretically passes through the assumed point of support and thus causes no shear.

Having developed an expression for the ordinate of the $x\Delta$ shear diagram at any load point between a and B, it is apparent that the area of the shear diagram may be expressed as the sum of the ordinates at the centers of the equal dx divisions multiplied by the length of those divisions. Hence the $x\Delta$ cantilever moment and the $y\Delta$ cantilever moment expressions, in both of which a, in the first summation symbol, is the point of load, are:

$$x\Delta \text{ cantilever moment} = \left(\sum_a^B \sum_{a+1}^B x\Delta\right) dx$$

$$y\Delta \text{ cantilever moment} = \left(\sum_a^B \sum_{a+1}^B y\Delta\right) dx$$

The above expressions are equated respectively to the terms

$$\sum_a^B (x-a)x\Delta \text{ and } \sum_a^B (x-a)y\Delta$$

and are substituted in equations (*7*) and (*8*), which now become:

$$H = \frac{\left(\sum_a^B \sum_{a+1}^B x\Delta \sum_A^B xy\Delta - \sum_a^B \sum_{a+1}^B y\Delta \sum_A^B x^2\Delta\right) dx}{\sum_A^B x^2\Delta \sum_A^B y^2\Delta - \left(\sum_A^B xy\Delta\right)^2} \quad (9)$$

$$V = \frac{\left(\sum_a^B \sum_{a+1}^B x\Delta \sum_A^B y^2\Delta - \sum_a^B \sum_{a+1}^B y\Delta \sum_A^B xy\Delta\right) dx}{\sum_A^B x^2\Delta \sum_A^B y^2\Delta - \left(\sum_A^B xy\Delta\right)^2} \quad (10)$$

For temperature change:

$$H_T = \pm \frac{(ETle)\left(\sum_A^B x^2\Delta\right)}{\sum_A^B x^2\Delta \sum_A^B y^2\Delta - \left(\sum_A^B xy\Delta\right)^2} \quad (11)$$

$$V_T = \pm \frac{(ETle)\left(\sum_A^B xy\Delta\right)}{\sum_A^B x^2\Delta \sum_A^B y^2\Delta - \left(\sum_A^B xy\Delta\right)^2} \quad (12)$$

in which

E = modulus of elasticity.
T = number of degrees change in temperature.
l = design span length.
e = coefficient of expansion.

It is customary to use the plus sign to designate values of H_T and V_T caused by a rise in temperature.

Equations (*9*), (*10*), (*11*), and (*12*) may now be used for evaluation of fixed-end moments and fixed-end thrusts due to concentrated loads and temperature change, as shown in the sample analysis.

Development of Arch Rib Design Constants

In figure 3 the hinge at A is assumed to be cut free, and joint B given a rotation α. Joint B is is then locked in this position and A is returned to its original position by first

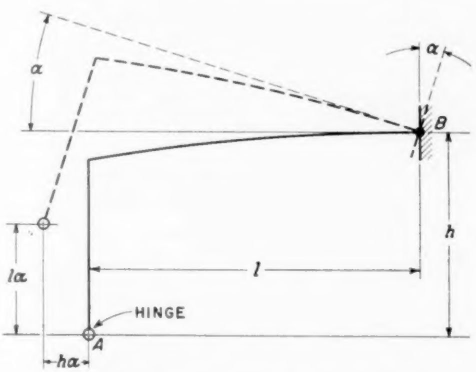


Figure 3.—Sketch for use in deriving moment stiffness of arch rib.

a horizontal displacement, $h\alpha$, without vertical displacement, and then by a vertical displacement, $l\alpha$, without horizontal displacement. The moment induced at B is that which would have been induced by holding A in its original position against displacement and rotating B through the angle α. The moment stiffness of the arch rib is expressed by the term M/α in which M is the moment at B induced by the rotation α.

The procedure is performed in two steps, as indicated in the previous paragraph. In step 1, A is given a horizontal displacement, $h\alpha$, without vertical displacement, and H_1, V_1, and M_{1B} are derived. In step 2, A is given a vertical displacement, $l\alpha$, without horizontal displacement, and H_2, V_2, and M_{2B} are derived. The final moment at B is thus $M_{1B}+M_{2B}$ and the final thrust is H_1+H_2. These correlated values may be used in obtaining the mathematical expressions for the previously defined arch rib design constants as follows:

Moment stiffness of arch rib $= (M_{1B}+M_{2B}) \div \alpha$
Induced thrust of arch rib $= (H_1-H_2) \div (M_{1B}+M_{2B})$
Thrust stiffness of arch rib $= H_1 + h\alpha$
Induced moment of arch rib $= M_{1B} \div H_1$

Step 1.—In deriving expressions for fixed-end moment and fixed-end thrust the modulus of elasticity, E, is omitted and the recurrent

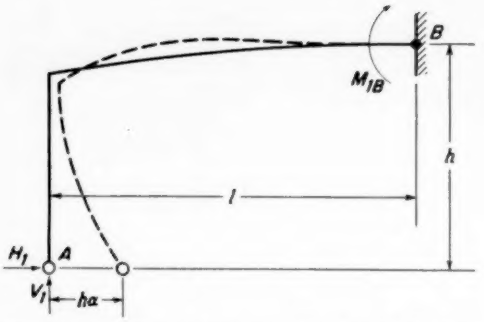


Figure 4.—Sketch for use in step 1 derivations.

term ds/I is represented by the symbol Δ for convenience. The absolute expressions for moment and thrust stiffness, however, contain E as a function. Therefore, the term Δ/E is used in deriving these constants.

From the condition that A is displaced horizontally a distance $h\alpha$ and from the condition that A is not displaced vertically, as shown in figure 4, the following two elastic equations may be written:

$$\sum_A^B My\frac{\Delta}{E}=h\alpha \qquad (13)$$

$$\sum_A^B Mx\frac{\Delta}{E}=0 \qquad (14)$$

No sign is affixed to the displacement term, $h\alpha$, since the direction of the displacement and of the forces causing it is obvious, as noted in figure 4.

The moment at any point between A and $B=V_1x-H_1y$. Inserting the general expression for moment into equations (*13*) and (*14*) the following are obtained:

$$V_1\sum_A^B xy\frac{\Delta}{E}-H_1\sum_A^B y^2\frac{\Delta}{E}=h\alpha \qquad (15)$$

$$V_1\sum_A^B x^2\frac{\Delta}{E}-H_1\sum_A^B xy\frac{\Delta}{E}=0 \qquad (16)$$

Solving equations (*15*) and (*16*) for H_1 and V_1:

$$H_1=\left[\frac{h\alpha\sum_A^B x^2\Delta}{\left(\sum_A^B xy\Delta\right)^2-\sum_A^B x^2\Delta\sum_A^B y^2\Delta}\right]E \qquad (17)$$

$$V_1=\left[\frac{h\alpha\sum_A^B xy\Delta}{\left(\sum_A^B xy\Delta\right)^2-\sum_A^B x^2\Delta\sum_A^B y^2\Delta}\right]E \qquad (18)$$

Whence:

$$M_{1B}=V_1l-H_1h=$$

$$\left[\frac{l\left(h\alpha\sum_A^B xy\Delta\right)-h\left(h\alpha\sum_A^B x^2\Delta\right)}{\left(\sum_A^B xy\Delta\right)^2-\sum_A^B x^2\Delta\sum_A^B y^2\Delta}\right]E \qquad (19)$$

Step 2.—From the condition that A is displaced a distance $l\alpha$ and from the condition that A is not displaced horizontally, as shown in figure 5, the following elastic equations may be written:

$$\sum_A^B Mx\frac{\Delta}{E}=l\alpha \qquad (20)$$

$$\sum_A^B My\frac{\Delta}{E}=0 \qquad (21)$$

The moment at any point between A and $B=H_2y-V_2x$. Inserting the general expression for moment into equations (*20*) and (*21*) the following are obtained:

$$H_2\sum_A^B xy\frac{\Delta}{E}-V_2\sum_A^B x^2\frac{\Delta}{E}=l\alpha \qquad (22)$$

$$H_2\sum_A^B y^2\frac{\Delta}{E}-V_2\sum_A^B xy\frac{\Delta}{E}=0 \qquad (23)$$

Solving equations (*22*) and (*23*) for H_2 and V_2:

$$H_2=\left[\frac{l\alpha\sum_A^B xy\Delta}{\left(\sum_A^B xy\Delta\right)^2-\sum_A^B x^2\Delta\sum_A^B y^2\Delta}\right]E \qquad (24)$$

$$V_2=\left[\frac{l\alpha\sum_A^B y^2\Delta}{\left(\sum_A^B xy\Delta\right)^2-\sum_A^B x^2\Delta\sum_A^B y^2\Delta}\right]E \qquad (25)$$

Whence:

$$M_{2B}=H_2h-V_2l=$$

$$\left[\frac{h\left(l\alpha\sum_A^B xy\Delta\right)-l\left(l\alpha\sum_A^B y^2\Delta\right)}{\left(\sum_A^B xy\Delta\right)^2-\sum_A^B x^2\Delta\sum_A^B y^2\Delta}\right]E \qquad (26)$$

The basic expressions for the arch rib design constants have already been stated. Substituting in these the expressions for H_1, V_1, and M_{1B}, derived as equations (*17*), (*18*), and (*19*) in step 1, and the expressions for H_2, V_2, and M_{2B}, derived as equations (*24*), (*25*), and (*26*) in step 2, the arch rib design constants may now be expressed as follows:

Moment stiffness of arch rib $=$
$(M_{1B}+M_{2B})\div\alpha=$

$$\left\{\left[\frac{l\left(h\alpha\sum_A^B xy\Delta-l\alpha\sum_A^B y^2\Delta\right)+h\left(l\alpha\sum_A^B xy\Delta-h\alpha\sum_A^B x^2\Delta\right)}{\left(\sum_A^B xy\Delta\right)^2-\sum_A^B x^2\Delta\sum_A^B y^2\Delta}\right]E\right\}\div\alpha=$$

$$\left[\frac{l\left(h\sum_A^B xy\Delta-l\sum_A^B y^2\Delta\right)+h\left(l\sum_A^B xy\Delta-h\sum_A^B x^2\Delta\right)}{\left(\sum_A^B xy\Delta\right)^2-\sum_A^B x^2\Delta\sum_A^B y^2\Delta}\right]E \qquad (27)$$

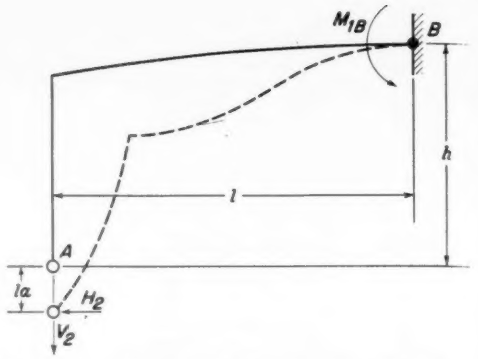


Figure 5.—Sketch for use in step 2 derivations.

Induced thrust of arch rib $=$
$(H_1-H_2)\div(M_{1B}+M_{2B})=$

$$\frac{h\sum_A^B x^2\Delta-l\sum_A^B xy\Delta}{l\left(h\sum_A^B xy\Delta-l\sum_A^B y^2\Delta\right)+h\left(l\sum_A^B xy\Delta-h\sum_A^B x^2\Delta\right)} \qquad (28)$$

Thrust stiffness of arch rib $=H_1\div h\alpha=$

$$\left[\frac{\sum_A^B x^2\Delta}{\left(\sum_A^B xy\Delta\right)^2-\sum_A^B x^2\Delta\sum_A^B y^2\Delta}\right]E \qquad (29)$$

Induced moment of arch rib $=M_{1B}\div H_1=$

$$\frac{l\sum_A^B xy\Delta-h\sum_A^B x^2\Delta}{\sum_A^B x^2\Delta} \qquad (30)$$

Development of Elastic Pier Constants

The elastic pier is assumed to be of uniform cross section, in accordance with common construction practice. The moment of inertia of this member is constant, therefore, and the expressions for moment stiffness, thrust stiffness, induced moment, and induced thrust can be derived directly without recourse to the summation process which is required in the case of the arch rib constants. Referring to figure 6:

Moment stiffness of pier $=$

$$\frac{M}{\alpha}=Hl\div\frac{l\alpha}{l}=\frac{(Hl)(3EI)}{Hl^2}=\frac{3EI}{l} \qquad (31)$$

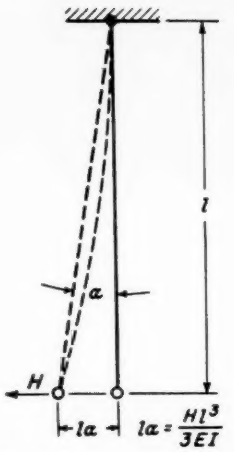


Figure 6.—Sketch for use in deriving pier constants.

Induced thrust of pier=

$$\frac{H}{M}=\frac{H}{Hl}=\frac{1}{l} \qquad (32)$$

Thrust stiffness of pier=

$$\frac{H}{l\alpha}=H\div\frac{Hl^3}{3EI}=\frac{3EI}{l^3} \qquad (33)$$

Induced moment of pier=

$$\frac{M}{H}=\frac{Hl}{H}=l \qquad (34)$$

Derivation of Sign Conventions

The system adopted for indicating the signs of the moments and thrusts is that which seems to be preferred by most designers. Moments tending to cause rotation about the joint in a clockwise direction are given a plus sign; moments which tend to cause rotation about the joint in a counter-clockwise direction are given a minus sign. Thrusts to the right are given a plus sign, and thrusts to the left are given a minus sign.

In figure 7 (a), joint *B* is locked and a load, *P*, is imposed on the arch. A moment is induced at *B*, and a thrust at *A*. The moment tends to rotate the joint in a counter-clockwise direction and is given a minus sign. The thrust is to the right and is given a plus sign.

Figure 7 (b) illustrates the action during the first moment distribution cycle. When fixity at *B* is removed, rotation about the joint is toward the imposed load; and the unbalanced fixed-end moment at *B* is stabilized by induced balancing moments in all of the members forming the joint. These balancing moments are proportional to the moment stiffnesses of the members, their sum exactly equals the fixed-end moment, and they all oppose the counter-clockwise rotation, thereby stabilizing the joint. They are accordingly given a plus sign.

The balancing moments induce thrusts as shown in figure 7 (b). Note that the fixed-end thrust has not been considered in the discussion, and that the thrusts shown in figure 7 (b) are entirely induced by the balancing moments.

These considerations lead to the establishment of the following rules of signs:

Arch ribs: Induced thrusts have the same sign as balancing moments. Induced moments have the same sign as balancing thrusts.

Pier: Induced thrusts have a sign opposite to that of balancing moments. Induced moments have a sign opposite to that of balancing thrusts.

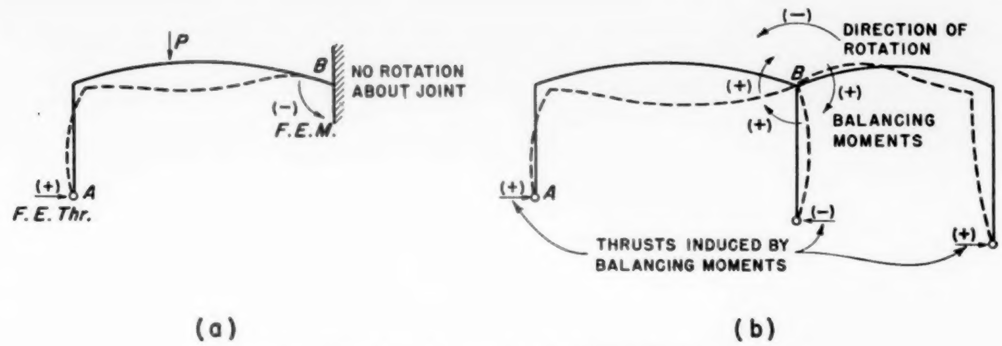


Figure 7.—Sketches for use in deriving sign convention.

SAMPLE ANALYSIS—I

Two-span arched frame with elastic pier, unsymmetrical both horizontally and vertically, and with hinged footings.

Application of Method

Evaluation of the design constants for arch ribs and pier of a two-span arched frame with elastic pier, unsymmetrical both horizontally and vertically, and with hinged footings, is illustrated in the sample analysis that follows.

Component terms in the expressions derived in the preceding section are entered in tabulation forms and are evaluated individually, and are then recombined to obtain the required final values. For greater clarity and speed, many component terms are referred to on the tabulation forms by column number.

Except in unusual cases the entire structure will be composed of the same structural material. The modulus of elasticity, *E*, may therefore be omitted in evaluating the stiffness constants, since only the relative stiffnesses are required.

(*Text continued on page 72.*)

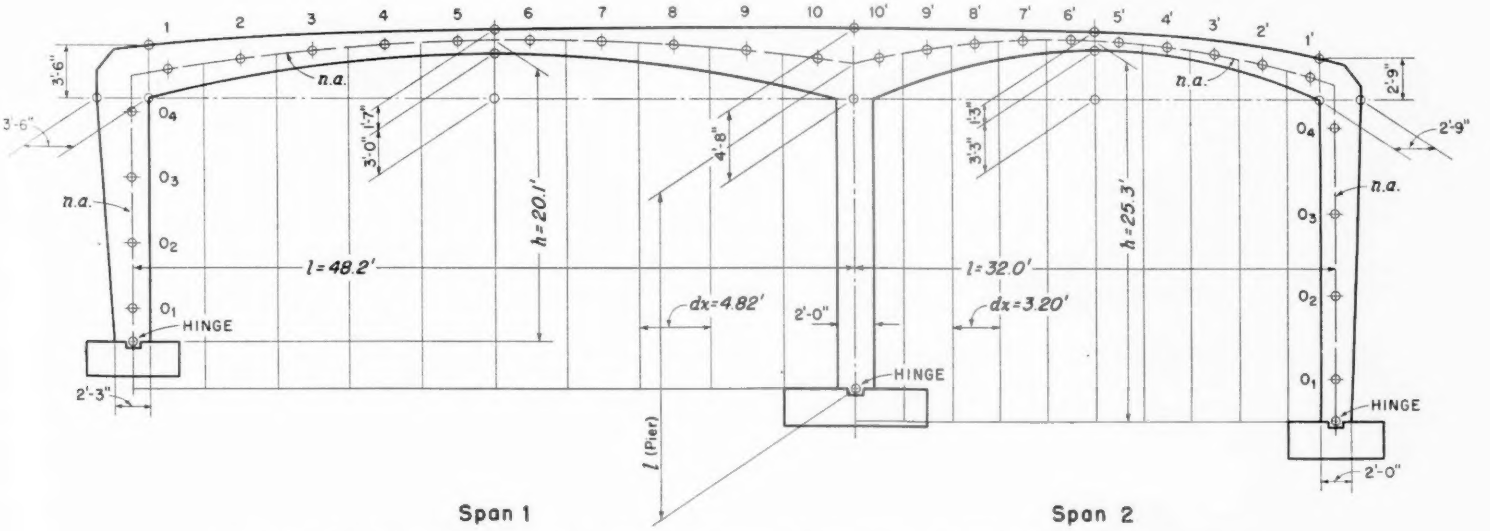


Figure 8.—Working drawing of hinged, unsymmetrical two-span arched frame with elastic pier.

Table 1 (a).—*Fixed-end moments, fixed-end thrusts, and joint constants, span 1*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Point or a	t	(col. 2)^3	ds	x	y	col. 4 ÷ col. 3	col. 5 × col. 7	col. 6 × col. 7	col. 5 × col. 8	col. 6 × col. 9	col. 5 × col. 9	$\sum_{a+1}^{B}$ col. 8	$\sum_{a+1}^{B}$ col. 9	$\sum_{a}^{B}$ col. 13	$\sum_{a}^{B}$ col. 14
0_1	2.46	14.887	4.87	0	2.43	0.327	0	0.80	0	2.6	0	0	0	0	0
0_2	2.78	21.485	4.87	0	7.30	.227	0	1.66	0	12.1	0	0	0	0	0
0_3	3.17	31.855	4.87	0	12.17	.153	0	1.86	0	22.6	0	0	0	0	0
0_4	3.45	41.064	4.87	0	17.04	.119	0	2.03	0	34.6	0	0	0	0	0
1	3.33	36.926	4.92	2.41	19.92	.133	.32	2.65	.8	52.8	6.4	130.42	119.55	650.01	520.24
2	2.58	17.174	4.88	7.23	20.58	.284	2.05	5.84	14.8	120.2	42.2	128.37	113.71	519.59	400.69
3	2.04	8.490	4.85	12.05	21.08	.571	6.88	12.04	82.9	253.8	145.1	121.49	101.67	391.22	286.98
4	1.70	4.913	4.81	16.87	21.42	.979	16.52	20.97	278.7	449.2	353.8	104.97	80.70	269.73	185.31
5	1.60	4.096	4.81	21.69	21.60	1.174	25.46	25.36	552.2	547.8	550.1	79.51	55.34	164.76	104.61
6	1.60	4.086	4.81	26.51	21.64	1.174	31.12	25.36	825.0	548.8	672.3	48.39	29.98	85.25	49.27
7	1.83	6.128	4.81	31.33	21.58	.785	24.59	16.94	770.4	365.6	530.7	23.80	13.04	36.86	19.29
8	2.33	12.649	4.85	36.15	21.33	.383	13.85	8.17	500.7	174.3	295.3	9.95	4.87	13.06	6.25
9	3.08	29.218	4.88	40.97	20.92	.167	6.84	3.49	280.2	73.0	143.0	3.11	1.38	3.11	1.38
10	4.17	72.512	4.92	45.79	20.35	.068	3.11	1.38	142.4	28.1	63.2	0	0	0	0
								$\sum$=	3,448	2,686	2,802				

1	17	18	19	20	21	22	23	24	25	26	27
Point or a	$\frac{\sum \text{col. 12}}{1{,}000}$ ×col. 15	$\frac{\sum \text{col. 10}}{1{,}000}$ ×col. 16	$\frac{\sum \text{col. 11}}{1{,}000}$ ×col. 15	$\frac{\sum \text{col. 12}}{1{,}000}$ ×col. 16	(col. 17−col. 18) ×dx	(col. 19−col. 20) ×dx	H= col. 21 ÷C	V= col. 22 ÷C	(col. 24 ×l)−(col. 23 ×h)	l−col. 5	M_B= col. 25−col. 26
0_1	0	0	0	0	0	0	0	0	0	0	0
0_2	0	0	0	0	0	0	0	0	0	0	0
0_3	0	0	0	0	0	0	0	0	0	0	0
0_4	0	0	0	0	0	0	0	0	0	0	0
1	1,821.3	1,793.8	1,745.9	1,457.7	132.5	1,389.1	.094	.985	45.59	45.79	−.20
2	1,455.9	1,381.6	1,395.6	1,122.7	358.1	1,315.4	.254	.933	39.86	40.97	−1.11
3	1,096.2	989.5	1,051.8	804.1	514.3	1,189.1	.365	.843	33.29	36.15	−2.86
4	755.8	638.9	724.5	519.2	563.5	989.5	.400	.702	25.80	31.33	−5.53
5	461.7	360.7	442.5	293.1	486.8	720.1	.345	.511	17.70	26.51	−8.81
6	238.9	169.9	229.0	138.1	332.6	438.1	.236	.311	10.25	21.69	−11.44
7	103.3	66.5	99.0	54.1	177.4	216.4	.126	.153	4.84	16.87	−12.03
8	36.6	21.6	35.1	17.5	72.3	84.8	.051	.060	1.82	12.05	−10.23
9	8.7	4.8	8.4	3.9	18.8	21.7	.013	.015	.46	7.23	−6.77
10	0	0	0	0	0	0	0	0	0	2.41	−2.41

h (arch rib) = 20.1 ft. l (arch rib) = 48.2 ft. dx = 4.82 ft.

$$C=\frac{\sum \text{col. 10} \sum \text{col. 11}-(\sum \text{col. 12})^2}{1{,}000}=1{,}410$$

$$\text{Moment stiffness of arch rib}=\frac{l\left(\frac{l\sum \text{col. 11}}{1{,}000}-\frac{h\sum \text{col. 12}}{1{,}000}\right)+h\left(\frac{h\sum \text{col. 10}}{1{,}000}-\frac{l\sum \text{col. 12}}{1{,}000}\right)}{12\,C}=0.130$$

$$\text{Induced thrust of arch rib}=\left(\frac{l\sum \text{col. 12}}{1{,}000}-\frac{h\sum \text{col. 10}}{1{,}000}\right)\div(\text{moment stiffness}\times 12\,C)=0.030$$

$$\text{Thrust stiffness of arch rib}=\frac{\sum \text{col. 10}}{1{,}000}\div 12\,C=0.00020$$

$$\text{Induced moment of arch rib}=\left(l\frac{\sum \text{col. 12}}{1{,}000}-h\frac{\sum \text{col. 10}}{1{,}000}\right)\div\frac{\sum \text{col. 10}}{1{,}000}=19.1$$

$$H_T=\frac{ETl\epsilon\times\frac{\sum \text{col. 10}}{1{,}000}}{12\,C}\qquad V_T=\frac{ETl\epsilon\times\frac{\sum \text{col. 12}}{1{,}000}}{12\,C}$$

Table 1 (b).—*Fixed-end moments, fixed-end thrusts, and joint constants, span 2*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Point or a	t	(col. 2) ^{3}	ds	x	y	col. 4 ÷ col. 3	col. 5 × col. 7	col. 6 × col. 7	col. 5 × col. 8	col. 6 × col. 9	col. 5 × col. 9	$\sum_{a+1}^{B}$ col. 8	$\sum_{a+1}^{B}$ col. 9	$\sum_{a}^{B}$ col. 13	$\sum_{a}^{B}$ col. 14
0_1	2.09	9.129	6.00	0	3.00	0.657	0	1.97	0	5.9	0	0	0	0	0
0_2	2.27	11.697	6.00	0	9.00	.513	0	4.62	0	41.6	0	0	0	0	0
0_3	2.45	14.706	6.00	0	15.00	.408	0	6.12	0	91.8	0	0	0	0	0
0_4	2.63	18.191	6.00	0	21.00	.330	0	6.93	0	145.5	0	0	0	0	0
1′	2.50	15.625	3.33	1.60	24.54	.213	.34	5.23	.5	128.3	8.4	86.93	159.53	405.93	645.28
2′	1.96	7.530	3.27	4.80	25.40	.434	2.08	11.02	10.0	279.9	52.9	84.85	148.51	319.00	485.75
3′	1.62	4.252	3.25	8.00	26.04	.764	6.11	19.89	48.9	517.9	159.1	78.74	118.62	234.15	337.24
4′	1.42	2.863	3.23	11.20	26.60	1.128	12.63	30.00	141.5	798.0	336.0	66.11	98.62	155.41	208.62
5′	1.35	2.460	3.21	14.40	26.88	1.305	18.79	35.08	270.6	943.0	505.2	47.32	63.54	89.30	110.00
6′	1.37	2.571	3.20	17.60	26.94	1.245	21.91	33.54	385.6	903.6	590.3	25.41	30.00	41.98	46.46
7′	1.67	4.657	3.22	20.80	26.83	.691	14.37	18.54	298.9	497.4	385.6	11.04	11.46	16.57	16.46
8′	2.25	11.391	3.24	24.00	26.62	.284	6.82	7.56	163.7	201.2	181.4	9.22	3.90	5.53	5.00
9′	3.12	30.371	3.26	27.20	26.18	.107	2.91	2.80	79.2	73.3	76.2	1.31	1.10	1.31	1.10
10′	4.25	76.766	3.29	30.40	25.60	.043	1.31	1.10	39.8	28.2	33.4	0	0	0	0
								Σ=	1,439	4,656	2,329				

1	17	18	19	20	21	22	23	24	25	26	27
Point or a	$\frac{\sum \text{col. } 12}{1{,}000}$ × col. 15	$\frac{\sum \text{col. } 10}{1{,}000}$ × col. 16	$\frac{\sum \text{col. } 11}{1{,}000}$ × col. 15	$\frac{\sum \text{col. } 12}{1{,}000}$ × col. 16	(col. 17 − col. 18) × dx	(col. 19 − col. 20) × dx	H= col. 21 ÷ C	V= col. 22 ÷ C	(col. 24 × l) − (col. 23 × h)	l − col. 5	$M_{B'}$= col. 26 − col. 25
0_1	0	0	0	0	0	0	0	0	0	0	0
0_2	0	0	0	0	0	0	0	0	0	0	0
0_3	0	0	0	0	0	0	0	0	0	0	0
0_4	0	0	0	0	0	0	0	0	0	0	0
1′	945.2	928.4	1,889.8	1,502.5	53.8	1,639.3	.042	.971	30.01	30.40	+.39
2′	742.8	698.8	1,485.1	1,132.8	140.8	1,132.8	.110	.888	25.64	27.20	+1.56
3′	545.2	485.2	1,090.1	785.3	192.0	975.4	.150	.764	20.65	24.00	+3.35
4′	361.9	300.1	723.5	485.8	197.8	760.6	.155	.596	15.15	20.80	+5.65
5′	207.9	158.3	415.7	256.1	158.7	510.7	.124	.400	9.66	17.60	+7.94
6′	97.8	66.8	195.4	108.2	99.2	279.0	.078	.219	5.04	14.40	+9.36
7′	38.6	23.7	77.1	38.3	47.7	124.2	.037	.097	2.16	11.20	+9.04
8′	12.9	7.2	25.7	11.6	18.2	45.1	.014	.035	.77	8.00	+7.23
9′	3.1	1.6	6.1	2.6	4.8	11.2	.004	.009	.19	4.80	+4.61
10′	0	0	0	0	0	0	0	0	0	1.60	+1.60

h (arch rib) = 25.3 ft. l (arch rib) = 32.0 ft. dx = 3.20 ft.

$$C=\frac{\sum \text{col. } 10 \sum \text{col. } 11-(\sum \text{col. } 12)^2}{1{,}000}=1{,}276$$

$$\text{Moment stiffness of arch rib}=\frac{l\left(\frac{l\sum \text{col. } 11}{1{,}000}-\frac{h\sum \text{col. } 12}{1{,}000}\right)+h\left(\frac{h\sum \text{col. } 10}{1{,}000}-\frac{l\sum \text{col. } 12}{1{,}000}\right)}{12\ C}=0.125$$

$$\text{Induced thrust of arch rib}=\left(\frac{l\sum \text{col. } 12}{1{,}000}-\frac{h\sum \text{col. } 10}{1{,}000}\right)\div(\text{moment stiffness}\times 12\ C)=0.020$$

$$\text{Thrust stiffness of arch rib}=\frac{\sum \text{col. } 10}{1{,}000}\div 12\ C=0.00009$$

$$\text{Induced moment of arch rib}=\left(l\frac{\sum \text{col. } 12}{1{,}000}-h\frac{\sum \text{col. } 10}{1{,}000}\right)\div\frac{\sum \text{col. } 10}{1{,}000}=26.5$$

$$H_T=\frac{ETle\times\frac{\sum \text{col. } 10}{1{,}000}}{12\ C} \qquad V_T=\frac{ETle\times\frac{\sum \text{col. } 12}{1{,}000}}{12\ C}$$

Table 2.—*Pier constants*

l=22.5 ft. t=2.0 ft.

$I=t^3/12=0.667$

Expression	Value
Moment stiffness $=3I\div l=$	0.089
Induced thrust $=1\div l=$	.044
Thrust stiffness $=3I\div l^3=$	.00018
Induced moment $=l=$	22.5

Table 3.—*Distribution of moment and thrust stiffness*

Member	Moment stiffness		Thrust stiffness	
	Value	Distribution factor	Value	Distribution factor
		Percent		*Percent*
Arch rib, span 1	0.130	38	0.00020	43
Arch rib, span 2	.125	36	.00009	19
Pier	.089	26	.00018	38
Total	0.344	100	0.00047	100

Table 4.—*Distribution of induced moments and thrusts*

Member	Induced moment	Induced thrust
Arch rib, span 1	19.1 × distributed thrusts	0.030 × distributed moments.
Arch rib, span 2	26.5 × distributed thrusts	0.020 × distributed moments.
Pier	22.5 × distributed thrusts	0.044 × distributed moments.

The structure (fig. 8) used for the sample analysis is unsymmetrical both horizontally and vertically. It has been chosen to emphasize the important advantage possessed by this method, in common with ordinary moment distribution, of being applicable to unsymmetrical as well as symmetrical structures with almost equal facility. In structures of this type the three frame footings are frequently at different elevations due to differences in the elevation of satisfactory foundation material. Horizontal dissymmetry has been less common in the past, but may be expected to occur more frequently in the construction of modern highways and interchanges, requiring numerous structures which must fit the alinements and clearances otherwise determined.

The sequence of the various steps in the sample analysis follows that used in the development of the method. The procedure as applied to an actual analysis is as follows:

1. A working drawing of the frames is made, and required basic data are scaled and entered in the tabulation forms.
2. The tabulation forms are then completed by the calculations indicated in the column headings, and the expressions below the tables are computed.
3. Moment and thrust distribution factors are computed.
4. A unit moment and a unit thrust are distributed individually at each side of the joint (at one side only if the structure is symmetrical).
5. Final distributed moments and thrusts are obtained by simple proportion to the unit distributed moments and thrusts.

1.—Construction and measurement of working drawing

The structural frames are laid out to a convenient scale, as shown in figure 8. Sufficient accuracy usually may be obtained with a scale of one-fourth or one-half inch equals one foot. The neutral axis lines of the arch ribs are drawn midway between the face surfaces. The neutral axis lines of the frame legs are drawn as perpendicular lines bisecting the bases of the legs at the footing tops. This involves a slight inaccuracy, but eliminates unwarranted refinement. The design spans between the neutral axis lines of the piers and frame legs are each divided into 10 equal horizontal parts which are projected vertically onto the neutral axis lines of the arch ribs, making 10 *ds* divisions.

The centers of gravity of the *ds* divisions are for convenience assumed to be at their midpoints in horizontal projection. These centers of gravity are numbered 1 through 10 for span 1 (the left arch) and 1′ through 10′ for span 2 (the right arch). Four longitudinal divisions are made of the frame leg neutral axis lines, and the midpoints of these *ds* divisions are located and designated O_1 through O_4. The lengths of the *ds* divisions are scaled and are entered in column 4 of the tabulation forms, tables 1 (a) and 1 (b), opposite the proper points. In the analysis, a load of unity is placed successively at each numbered point on the arch ribs.

Values of t, x, and y at each load point are obtained by scaling, and are recorded in columns 2, 5, and 6 on the tabulation forms. The required x and y values are, respectively, the horizontal distance from the neutral axis line of the frame leg and the vertical distances from a level line through the hinge, to the numbered load points. The t values are the thicknesses of the various sections, measured radially through each load point. The remaining data for the analysis are derived in the tabulation from these basic measurements.

2.—Completion of tabulation forms

Moments of inertia of the sections are computed in column 3 as t^3 instead of the true value, $t^3 \div 12$, in order to avoid large figures in the subsequent columns. It is necessary, however, to reinsert the factor 12 in some of the final expressions in order to make them applicable to a section 1 foot wide instead of 12 feet wide, and this is done in the stiffness and thrust expressions and also in the expressions for H_t and V_t, which appear below the tabulations in tables 1 (a) and 1 (b).

Entries for columns 7 to 12, inclusive, are computed as indicated by the column heads. Note that totals are recorded for columns 10, 11, and 12.

The method of computing entries for columns 13 to 16, inclusive, involves the summation process, and an explanation of this procedure may be helpful. Assume that it is desired to solve for H and M_B due to a load of unity at point 1. Point 1 is therefore taken as a. The entry for column 13 is to be the summation from $a+1$ to B of $x\Delta$, which is the sum of the $x\Delta$ values in column 8 from point 2 to point 10, inclusive. The entry for column 14 is similarly computed except that $y\Delta$ values, in column 9, are used for summation.

The entry in column 15 is to be the summation from a to B of column 13, and for point 1 this is the sum of the values in column 13 from point 1 to point 10, inclusive. The entry for column 16 is similarly computed except that values in column 14 are used for summation.

If H and M_B were desired only for a load of unity at point 1, it would simply be necessary to complete the operations indicated in columns 17 to 27, inclusive, along the line opposite point 1. In the sample analysis H and M_B are computed for 10 positions of the unit load on each arch, and the tabulations are completed in full.

In several of the columns of the tabulation forms, tables 1 (a) and 1 (b), and in the expressions below them, division by 1,000 is indicated. This is merely a device to avoid large figures, not a function of the basic formulas.

The constant C, which appears below the tabulation forms, is a term that is derived from the totals of columns 10, 11, and 12. It is evaluated independently for convenience in use in the computation of entries for columns 23 and 24 and for the expressions which appear below the form.

The computation of the pier constants is illustrated in table 2 (bottom of page 71).

3.—Computation of moment and thrust distribution factors

The values derived in the expressions below the tabulation forms in tables 1 (a) and 1 (b), and in table 2, are used in obtaining relative values of moment and thrust stiffness, and values of induced moments and thrusts.

The moment and thrust distribution factors are evaluated in exactly the same manner as in ordinary moment distribution. This is illustrated in table 3. The moment stiffness values for the structure members, derived in tables 1 (a), 1 (b), and 2, are entered in the proper column and are totaled. Each value is divided by the total, yielding the distribution factor. The thrust stiffness distribution is handled in the same manner.

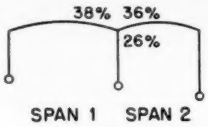
The method of distributing the induced moments and thrusts is shown in table 4, for which the figures were derived in tables 1 (a), 1 (b), and 2. The values are constants by which distributed moments and distributed thrusts are multiplied, for purposes described in the next step of the procedure.

4.—Distribution of a unit moment and a unit thrust

Completion of the tabulation forms, tables 1 (a) and 1 (b), provides the fixed-end moments and their correlated fixed-end thrusts due to a unit load at 10 positions on each arch. The juncture of the arch ribs and pier has thus far been considered completely fixed, so that transfer of moment or thrust from one member to the others is not permitted. The next step consists of distributing each correlated fixed-end moment and fixed-end thrust so that for each position of the unit load actual moments at the joint and reactions at all three footings are obtained.

Tables 5–8.—*Unit moment and thrust distributions*

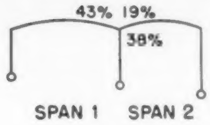
MOMENT DISTRIBUTION FACTORS



INDUCED THRUSTS

0.030 × moment (arch rib, span 1)
0.020 × moment (arch rib, span 2)
0.044 × moment (pier)

THRUST DISTRIBUTION FACTORS



INDUCED MOMENTS

19.1 × thrust (arch rib, span 1)
26.5 × thrust (arch rib, span 2)
22.5 × thrust (pier)

Table 5.—*Distribution of unit moment, span 1*

	Rib	Pier	Rib		Rib	Pier	Rib	
Fixed-end moment	−1.000	0	0		0	0	0	Fixed-end thrust
Balancing moment	+.380	+.260	+.360	Induced thrust→	+.011	−.011	+.007	
	−.057	+.068	−.027	←Induced moment	−.003	−.003	−.001	Balancing thrust
Balancing moment	+.006	+.004	+.006	Induced thrust→	----	----	----	
	----	----	----	←Induced moment	----	----	----	Balancing thrust
Balancing moment	----	----	----		+.008	−.014	+.006	Final thrusts
Final moments	−.671	+.332	+.339					

Table 6.—*Distribution of unit thrust, span 1*

	Rib	Pier	Rib		Rib	Pier	Rib	
Fixed-end moment	0	0	0		+1.000	0	0	Fixed-end thrust
	−8.213	+8.550	−5.035	←Induced moment	−.430	−.380	−.190	Balancing thrust
Balancing moment	+1.785	+1.221	+1.691	Induced thrust→	+.054	−.054	+.034	
	−.287	+.293	−.159	←Induced moment	−.015	−.013	−.006	Balancing thrust
Balancing moment	+.058	+.040	+.055	Induced thrust→	+.002	−.002	+.001	
Final moments	−6.657	+10.104	−3.448		------	------	------	Balancing thrust
					+.611	−.449	−.161	Final thrusts

Table 7.—*Distribution of unit moment, span 2*

	Rib	Pier	Rib		Rib	Pier	Rib	
Fixed-end moment	0	0	+1.000		0	0	0	Fixed-end thrust
Balancing moment	−.380	−.260	−.360	Induced thrust→	−.011	+.011	−.007	
	+.057	−.068	+.027	←Induced moment	+.003	+.003	+.001	Balancing thrust
Balancing moment	−.006	−.004	−.006	Induced thrust→	------	------	------	
	------	------	------	←Induced moment	------	------	------	Balancing thrust
Balancing moment	------	------	------		−.008	+.014	−.006	Final thrusts
Final moments	−.329	−.332	+.661					

Table 8.—*Distribution of unit thrust, span 2*

	Rib	Pier	Rib		Rib	Pier	Rib	
Fixed-end moment	0	0	0		0	0	−1.000	Fixed-end thrust
	+8.213	−8.550	+5.035	←Induced moment	+.430	+.380	+.190	Balancing thrust
Balancing moment	−1.785	−1.221	−1.691	Induced thrust→	−.054	+.054	−.034	
	+.287	−.293	+.159	←Induced moment	+.015	+.013	+.006	Balancing thrust
Balancing moment	−.058	−.040	−.055	Induced thrust→	−.002	+.002	−.001	
	------	------	------		------	------	------	Balancing thrust
Final moments	+6.657	−10.104	+3.448		+.389	+.449	−.839	Final thrusts

Since 20 positions of the unit load on both arches are considered, it would appear that a prohibitive number of distributions is required. Actually, only two distributions are required for symmetrical structures, and four if the structure is unsymmetrical.

Referring to table 5 in the sample analysis, it will be noted that a fixed-end moment of unity is applied to the joint at the juncture of the left arch rib. This moment is distributed in a manner similar to that of ordinary moment distribution, using the moment distribution factors computed in table 3 and shown around the joint in the left-hand sketch above table 5. The values of the distributed moments are shown in the left tabulation of table 5. This shows, first, the unit moment of −1.000 opposite the stub "fixed-end moment," and, immediately below, the distributing moments opposite the stub "balancing moments."

Next, the first arrow, "induced thrusts," is followed to the tabulation at the right, and the thrusts induced by the balancing moments are entered. These values are obtained by multiplying each balancing moment by the corresponding induced thrust constants, previously computed in table 4 and repeated for convenience under the sketch above table 5.

Next, these unbalanced thrusts are balanced exactly as if they were unbalanced moments, but using the thrust distribution factors computed in table 3 and shown around the joint in the right-hand sketch above table 5. Now the second arrow, "induced moments," is followed to the left, and the new moments induced by the balancing thrusts are entered. These values are obtained by multiplying each balancing thrust by the corresponding induced moment constants, previously computed in table 4 and repeated for convenience under the sketch above table 5.

This procedure is repeated until the converging values become so small that further refinement is unnecessary. As shown in the actual example, convergence occurs rapidly.

In table 6 a fixed-end thrust of unity is distributed in the same manner as described above for a unit moment, and final values of moments and thrusts are obtained.

The significance of these procedures may be summarized as follows:

(*a*) A load is placed on one of the frames but the joint at the middle is fixed, so that the moment at the joint and the outward "kick," or thrust, are so-called "fixed-end" values.

(*b*) Considering the fixed-end moment and fixed-end thrust independently of each other, the fixity at the joint is first released, and then final values of moment and thrust due only to the fixed-end moment are computed. Computation is similarly made of final values of moment and thrust due only to the fixed-end thrust.

(*c*) Individual values are now added algebraically to obtain final values due to the load that caused the fixed-end moment and fixed-end thrust.

The structure analyzed in the example is unsymmetrical, and the process is therefore repeated for the right arch in tables 7 and 8. It is important to note, however, that all the distributed values in the distribution for the

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right arch are identical with those for the left arch, with signs changed. This fact saves considerable time in the second procedure.

5.—Evaluation of final distributed moments and thrusts

The manner of obtaining final distributed moments and thrusts for span 1 is illustrated in tables 9 (a), 9 (b), and 9 (c) of the sample analysis. In table 9 (a) are recorded the values of M_B, $M_{B'}$, H_1, H_2, and H_3 due to a distributed fixed-end moment of unity; and similar values due to a distributed fixed-end thrust of unity are recorded in table 9 (b). These values are, of course, obtained from tables 5 and 6.

In the second column of table 9 (c) the fixed-end moments at each load point are recorded as obtained in column 27 of table 1 (a). In the third column the fixed-end thrusts at each load point are recorded as obtained in column 23 of table 1 (a).

Each of the values in table 9 (a) is then multiplied by the actual fixed-end moment for each load point and the results recorded. Similarly, each of the values in table 9 (b) is multiplied by the actual fixed-end thrust for each load point. The sums of the corresponding pairs of values obtained by these two series of multiplications are then recorded, and are the final distributed values of moment and thrust for a load of unity at each load point.

The final distributed moments and thrusts for span 2 are obtained in the same manner, as illustrated in tables 10 (a), 10 (b), and 10 (c) of the sample analysis.

Table 9.—*Tabulation of final moments and thrusts, span 1*

Table 9 (a)

Fixed-end moment = −1.000				
M_B	$M_{B'}$	H_1	H_2	H_3
−0.671	+0.339	+0.008	−0.014	+0.006

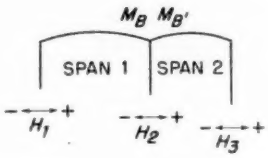


Table 9 (b)

Fixed-end thrust = +1.000				
M_B	$M_{B'}$	H_1	H_2	H_3
−6.657	−3.448	+0.611	−0 449	−0.161

Table 9 (c)

Unit load at point—	Fixed-end moment	Fixed-end thrust	M and H due to fixed-end moment					M and H due to fixed-end thrust					Final values of M and H				
			M_B	$M_{B'}$	H_1	H_2	H_3	M_B	$M_{B'}$	H_1	H_2	H_3	M_B	$M_{B'}$	H_1	H_2	H_3
1	−0.20	+0.094	−0.13	+0.07	+0.002	−0.003	+0.001	−0.62	−0.32	+0.058	−0.042	−0.015	−0.75	−0.25	+0.060	−0.045	−0.014
2	−1.11	+.254	−.72	+.38	+.010	−.017	+.007	−1.68	−.87	+.156	−.115	−.041	−2.40	−.49	+.166	−.132	−.037
3	−2.86	+.365	−1.86	+.97	+.026	−.043	+.017	−2.41	−1.25	+.224	−.165	−.058	−4.27	−.28	+.250	−.208	−.041
4	−5.53	+.400	−3.59	+1.88	+.050	−.083	+.033	−2.65	−1.37	+.245	−.181	−.064	−6.24	+.51	+.295	−.264	−.031
5	−8.81	+.345	−5.73	+3.00	+.079	−.132	+.053	−2.28	−1.19	+.211	−.156	−.055	−8.01	+1.81	+.290	−.288	−.002
6	−11.44	+.236	−7.44	+3.89	+.102	−.172	+.069	−1.56	−.81	+.145	−.107	−.038	−9.00	+3.08	+.247	−.279	+.031
7	−12.03	+.126	−7.82	+4.09	+.108	−.180	+.072	−.83	−.43	+.077	−.057	−.020	−8.65	+3.66	+.185	−.237	+.052
8	−10.23	+.051	−6.65	+3.48	+.092	−.153	+.061	−.34	−.18	+.031	−.023	−.008	−6.99	+3.30	+.123	−.176	+.053
9	−6.77	+.013	−4.40	+2.30	+.061	−.102	+.041	−.09	−.04	+.008	−.006	−.002	−4.49	+2.26	+.069	−.108	+.039
10	−2.41	0	−1.57	+.82	+.022	−.036	+.014	0	0	0	0	0	−1.57	+.82	+.022	−.036	+.014

Table 10.—*Tabulation of final moments and thrusts, span 2*

Table 10 (a)

Fixed-end moment = +1.000				
M_B	$M_{B'}$	H_1	H_2	H_3
−0.329	+0.661	−0.008	+0.014	−0.006

Table 10 (b)

Fixed-end thrust = −1.000				
M_B	$M_{B'}$	H_1	H_2	H_3
+6.657	+3.448	+0.389	+0.449	−0.839

Table 10 (c)

Unit load at point—	Fixed-end moment	Fixed-end thrust	M and H due to fixed-end moment					M and H due to fixed-end thrust					Final values of M and H				
			M_B	$M_{B'}$	H_1	H_2	H_3	M_B	$M_{B'}$	H_1	H_2	H_3	M_B	$M_{B'}$	H_1	H_2	H_3
10′	+1.60	0	−0.56	+1.06	−0.014	+0.024	−0.010	0	0	0	0	0	−0.56	+1.06	−0.014	+0.024	−0.010
9′	+4.61	−.004	−1.61	+3.04	−.041	+.069	−.028	+.026	+.014	+.002	+.002	−.003	−1.58	+3.05	−.039	+.071	−.031
8′	+7.23	−.014	−2.53	+4.77	−.065	+.108	−.043	+.093	+.048	+.005	+.006	−.012	−2.44	+4.82	−.060	+.114	−.055
7′	+9.04	−.037	−3.16	+5.97	−.081	+.136	−.054	+.245	+.127	+.014	+.017	−.031	−2.91	+6.10	−.067	+.153	−.085
6′	+9.36	−.078	−3.28	+6.18	−.084	+.140	−.056	+.516	+.268	+.030	+.035	−.066	−2.76	+6.45	−.054	+.175	−.122
5′	+7.94	−.124	−2.78	+5.24	−.071	+.119	−.048	+.820	+.426	+.048	+.056	−.104	−1.96	+5.67	−.023	+.175	−.152
4′	+5.65	−.155	−1.98	+3.73	−.051	+.085	−.034	+1.025	+.533	+.060	+.070	−.130	−0.95	+4.26	+.009	+.155	−.164
3′	+3.35	−.150	−1.17	+2.21	−.030	+.050	−.020	+.992	+.515	+.058	+.068	−.126	−0.18	+2.73	+.028	+.118	+.146
2′	+1.56	−.110	−.55	+1.03	−.014	+.023	−.069	+.728	+.378	+.043	+.050	−.092	+0.18	+1.41	+.029	+.073	−.101
1′	+.39	−.042	−.14	+.26	−.004	+.006	−.002	+.278	+.144	+.016	+.019	−.035	+0.14	+.40	+.012	+.025	−.037

Required Design Constants

The joint constants for the condition of fixed footings include all of the design constants used in the analysis for hinged footings with identical definitions (see page 66). To this group are added the following:

Induced moment at footing.—The moment induced at the footing by a horizontal thrust at B without vertical displacement or rotation at B.

Moment carry-over.—The ratio of moment induced at the footing to an applied moment at B without horizontal or vertical displacement at B.

Development of Fixed-End Moment and Fixed-End Thrust

From the conditions that point A is not displaced horizontally, that point A is not displaced vertically, and that no rotation of

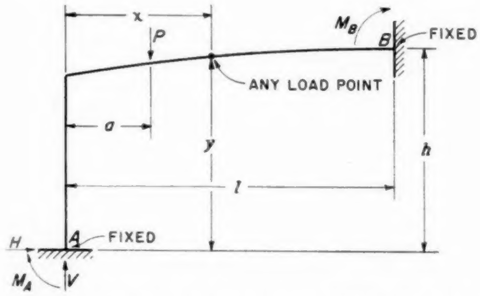


Figure 9.—Sketch for use in deriving fixed-end moments and thrusts.

the end tangents occurs, as shown in figure 9, the following elastic equations may be written:

$$\sum_A^B M\frac{ds}{EI} \text{ (change in angle between end tangents)}=0 \qquad (35)$$

$$\sum_A^B Mx\frac{ds}{EI} \text{ (vertical displacement of } A)=0 \qquad (36)$$

$$\sum_A^B My\frac{ds}{EI} \text{ (horizontal displacement of } A)=0 \qquad (37)$$

The moment at any point between A and $a = M_A + Vx - Hy$.

The moment at any point between a and B, when P is unity, $= M_A + Vx - Hy - (x-a)$.

For convenience, the following symbols are adopted:

$$\frac{ds}{I}=\Delta \qquad \sum_A^B x\Delta = B \qquad \sum_A^B x^2\Delta = D$$

$$\sum_A^B \Delta = A \qquad \sum_A^B y\Delta = C \qquad \sum_A^B y^2\Delta = E$$

$$\sum_A^B xy\Delta = F \qquad \sum_a^B (x-a)x\Delta = K_2$$

$$\sum_a^B (x-a)\Delta = K_1 \qquad \sum_a^B (x-a)y\Delta = K_3$$

E, being constant for the entire structure, may be eliminated from the basic equations in evaluating external moments and reactions due to flexure.

Inserting the general expression for moment in equations (35), (36), and (37), and using the abbreviated notation, the following equations are obtained:

$$M_A A + VB - HC - K_1 = 0 \qquad (38)$$

$$M_A B + VD - HF - K_2 = 0 \qquad (39)$$

$$M_A C + VF - HE - K_3 = 0 \qquad (40)$$

Solving equations (38) and (39):

$$V\left(\frac{B^2}{A}-D\right)-H\left(\frac{CB}{A}-F\right)-K_1\frac{B}{A}+K_2=0 \qquad (41)$$

Solving equations (39) and (40):

$$V\left(\frac{DC}{B}-F\right)-H\left(\frac{FC}{B}-E\right)-K_2\frac{C}{B}+K_3=0 \qquad (42)$$

Solving equations (41) and (42) for H, and clearing:

$$H=\frac{K_3\left(\frac{B^2}{A}-D\right)-K_2\left[\frac{C}{B}\left(\frac{B^2}{A}-D\right)+\left(\frac{DC}{B}-F\right)\right]+K_1\frac{B}{A}\left(\frac{DC}{B}-F\right)}{\left(\frac{B^2}{A}-D\right)\left(\frac{FC}{B}-E\right)-\left(\frac{CB}{A}-F\right)\left(\frac{DC}{B}-F\right)} \qquad (43)$$

For further simplification the following symbols are used:

$$C_1=\frac{C}{B} \qquad C_6=\frac{CB}{A}-F$$

$$C_2=\frac{B}{A} \qquad C_7=\frac{D}{B}$$

$$C_3=\frac{B^2}{A}-D \qquad C_8=\frac{AD}{B}-B$$

$$C_4=\frac{DC}{B}-F \qquad C_H=C_3C_5-C_6C_4$$

$$C_5=\frac{FC}{B}-E$$

Substituting these symbols in equation (43):

$$H=\frac{K_1C_2C_4-K_2C_6+K_3C_3}{C_H} \qquad (44)$$

Substituting the symbols in equation (41):

$$V=\frac{HC_6+K_1C_2-K_2}{C_3} \qquad (45)$$

From equations (38) and (39):

$$M_A=\frac{HC_4+K_1C_7-K_2}{C_8} \qquad (46)$$

The expressions for M_A, H, and V are evaluated by means of the tabular computation form, illustrated in the sample analysis. As in the tabular form for hinged footings, the expressions

$$\sum_a^B (x-a)\Delta,\ \sum_A^B (x-a)x\Delta, \text{ and } \sum_a^B (x-a)y\Delta$$

are equated to the expressions

$$\sum_a^B\sum_{a+1}^B \Delta dx,\ \sum_a^B\sum_{a+1}^B x\Delta dx, \text{ and } \sum_a^B\sum_{a+1}^B y\Delta dx$$

The general procedure in adapting the various expressions to a tabular computation form is also similar to that employed in the analysis for hinged footings, and reference thereto will be helpful in studying the form as arranged for fixed footings.

Development of Arch Rib Design Constants

The arch rib design constants are derived in two steps. First the rib is given a unit rotation at B, and H_1, V_1, M_{1A}, and M_{1B} are computed. No horizontal or vertical displacement is permitted in this step. Then the rib is given a unit horizontal displacement without vertical displacement, and H_2, V_2, M_{2A}, and M_{2B} are computed. These values of M, V, and H are then combined as required to obtain all the necessary joint constants, as well as expressions for H_T and V_T caused by temperature change.

In deriving the expressions, the system of notation used in obtaining expressions for fixed-end moments and thrusts is adopted. Values obtained in the computation for fixed-end moments and thrusts are used in evaluating the expressions for the joint constants. No additional basic computation is required.

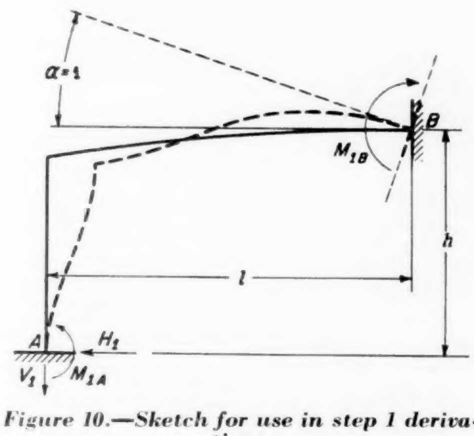


Figure 10.—Sketch for use in step 1 derivations.

Step 1.—The arch rib is given a unit rotation at B and no horizontal or vertical displacement at A is permitted, as shown in figure 10. Then:

$$M_{1A}A+V_1B-H_1C=1 \qquad (47)$$

$$M_{1A}B+V_1D-H_1F=l \qquad (48)$$

$$M_{1A}C+V_1F-H_1E=h \qquad (49)$$

Solving equations (47) and (48):

$$V_1\left(\frac{B^2}{A}-D\right)-H_1\left(\frac{CB}{A}-F\right)=\frac{B}{A}-l \text{ - - } (50)$$

Solving equations (48) and (49):

$$V_1\left(\frac{DC}{B}-F\right)-H_1\left(\frac{FC}{B}-E\right)=l\frac{C}{B}-h \text{ - - } (51)$$

The coefficients 10, 100, and 10,000 are introduced in the following equations to make the expressions for H_1 V_1, and M_{1A} applicable to the computation form used in the sample analysis, in which actual values of x and y are divided by 10, reducing the numerical order of the derived quantities.

Solving equations (50) and (51), and clearing:

$$H_1=\frac{(lC_1-h)100C_3-(10C_2-l)100C_4}{-10{,}000C_H} \text{ - - - } (52)$$

From equation (50):

$$V_1=\frac{H_1(100C_6)+10C_2-l}{100C_3} \text{ - - - - - - } (53)$$

From equations (47) and (48):

$$M_{1A}=\frac{H_1(100\ C_4)+10\ C_7-l}{10\ C_8} \text{ - - - - - } (54)$$

From statics:

$$M_{1B}=M_{1A}+V_1l-H_1h \text{ - - - - - - - } (55)$$

Step 2.—The arch rib is given a unit horizontal displacement, without vertical displacement, equal to the arch height, h, as shown in figure 11. Then:

$$M_{2A}C+VF-HE=h \text{ - - - - - - - } (56)$$

$$M_{2A}B+VD-HF=0 \text{ - - - - - - - } (57)$$

$$M_{2A}A+VB-HC=0 \text{ - - - - - - - } (58)$$

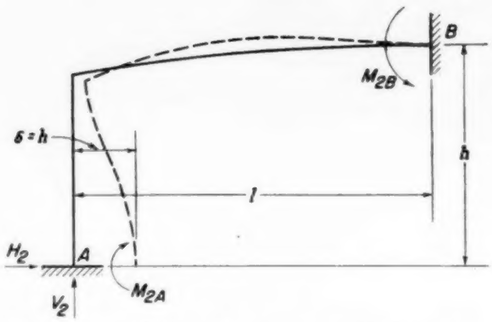


Figure 11.—Sketch for use in step 2 derivations.

Solving equations (56) and (57):

$$V\left(\frac{DC}{B}-F\right)-H\left(\frac{FC}{B}-E\right)=-h \text{ - - - } (59)$$

Solving equations (57) and (58):

$$V\left(\frac{B^2}{A}-D\right)-H\left(\frac{CB}{A}-F\right)=0 \text{ - - - - } (60)$$

Solving equations (59) and (60), and clearing:

$$H_2=\frac{-h(100C_3)}{-1{,}000C_H} \text{ - - - - - - - - - - } (61)$$

From equation (60):

$$V_2=\frac{H_2C_6}{C_3} \text{ - - - - - - - - - - - - - } (62)$$

Equations (61) and (62) may be used in evaluating H_T and V_T due to temperature change by substituting $\pm$ $(ETle)$ $\div$ 12 for h. The factor 12 is inserted to correct for the use of t^3 instead of $t^3/12$ (where $t=$ radial depths of sections at load points) for values of I in the computation form.

Solving equations (57) and (58) for M_{2A}:

$$M_{2A}=\frac{H_2(100C_4)}{10C_8} \text{ - - - - - - - - - - } (63)$$

From statics:

$$M_{2B}=M_{2A}+V_2l-H_2h \text{ - - - - - - - - } (64)$$

Using the values of H_1, V_1, M_{1A}, M_{2A}, H_2, V_2, M_{2A}, and M_{2B}, and inserting the factor 12 where required to correct for the use of t^3 instead of $t^3/12$ in the computation form, the joint constants for the arch ribs may be expressed as follows:

Moment stiffness $=M_{1B}\div 12$

Induced thrust $=H_1\div M_{1B}$

Thrust stiffness $=H_2\div 12\ h$

Induced moment (at pier top) $=M_{2B}\div H_2$

Induced moment (at footing) $=-M_{2A}\div H_2$

Moment carry-over $=-M_{1A}\div M_{1B}$

Development of Elastic Pier Constants

The elastic pier constants are derived (see fig. 12) by giving A a unit rotation without vertical displacement, and evaluating M_A, the moment stiffness, and P/M, the induced thrust. P is the force developed in restraining the ends against displacement. Moment carry-over is expressed by the term, M_B/M_A.

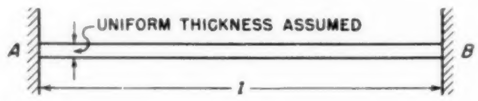


Figure 12.—Sketch for use in deriving elastic pier constants.

Point A is then given a vertical displacement, $\delta=1$, from which the thrust stiffness, P/δ, and induced moment, M_A/P, are evaluated.

Performing these operations, the following expressions for the elastic pier constants are obtained:

Moment stiffness $=4\ I\div l$ (relative)
Induced thrust $=3\div 2l$
Thrust stiffness $=12\ I\div l^3$ (relative)
Induced moment $=l\div 2$
Moment carry-over $=+0.5$

SAMPLE ANALYSIS—II

Two-span arched frame with elastic pier, unsymmetrical both horizontally and vertically, and with fixed footings.

Application of Method

Evaluation of the design constants for arch ribs and pier of a two-span arched frame with elastic pier, unsymmetrical both horizontally and vertically, and with fixed footings, is illustrated in the sample analysis that follows. The structure is identical with that analyzed in part I (fig. 8), except that the footings are fixed instead of hinged.

The general procedure and sign convention are closely similar to those used in the analysis of the structure with hinged footings. A working drawing of the frames is made to convenient scale, from which values of t, ds, x, and y are scaled. These values are entered in tables 11 (a) and 11 (b), the scaled x and y dimensions first being divided by 10.

The tables are then completed by the computations indicated in the column headings. It will be noted that columns 7 to 12, inclusive, are each totaled to obtain values of A, B, C, D, E, and F, which in turn are used to obtain the C "subscript" series of values. The latter are used in computation of entries in some of the table columns.

The C "subscript" values are also used in computing the moments, vertical reactions, and thrusts for each span, as shown in table 11 (c). These in turn are used to derive arch rib joint constants as illustrated in table 11 (d). Derivation of the pier constants appears in table 12.

Moment and thrust stiffness distribution factors are computed in table 13, and the method of distributing induced moments and thrusts and moment carry-overs is shown in table 14. Tables 15, 16, 17, and 18 illustrate the distribution of a unit moment and a unit thrust for each of the two spans. The evaluation of final distributed moments and thrusts is shown in tables 19 and 20.

Additional computations are provided in tables 21 and 22 for obtaining moments induced at the footing by balancing thrusts, and moments carried over by balancing moments.

Table 11 (a).—*Fixed-end moments and fixed-end thrusts, span 1*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Point or a	t	(col. 2) ^{3}	ds	$\frac{x}{10}$	$\frac{y}{10}$	col. 4 ÷ col. 3	col. 5 × col. 7	col. 6 × col. 7	col. 5 × col. 8	col. 6 × col. 9	col. 5 × col. 9	$\sum_{a+1}^{B}$ col. 7	$\sum_{a+1}^{B}$ col. 8	$\sum_{a+1}^{B}$ col. 9	$\sum_{a}^{B}$ col. 13 ×0.1dx	$\sum_{a}^{B}$ col. 14 ×0.1dx	$\sum_{a}^{B}$ col. 15 ×0.1dx
0_1	2.46	14.887	4.87	0	0.243	0.327	0	0.080	0	0.019	0	0	0	0	0	0	0
0^2	2.78	21.485	4.87	0	.730	.227	0	.166	0	.121	0	0	0	0	0	0	0
0_3	3.17	31.855	4.87	0	1.217	.153	0	.186	0	.226	0	0	0	0	0	0	0
0_4	3.45	41.064	4.87	0	1.704	.119	0	.203	0	.346	0	0	0	0	0	0	0
1	3.33	36.926	4.92	.241	1.992	.133	.032	.265	.008	.528	.064	5.585	13.042	11.960	11.697	31.330	25.088
2	2.58	17.174	4.88	.723	2.058	.284	.205	.584	.148	1.202	.422	5.301	12.837	11.376	9.005	25.044	19.323
3	2.04	8.490	4.85	1.205	2.108	.571	.688	1.204	.829	2.538	1.451	4.730	12.149	10.172	6.450	18.857	13.840
4	1.70	4.913	4.81	1.687	2.142	.979	1.652	2.097	2.787	4.492	3.538	3.751	10.497	8.075	4.170	13.001	8.937
5	1.60	4.096	4.81	2.169	2.160	1.174	2.546	2.536	5.522	5.478	5.501	2.577	7.951	5.539	2.362	7.941	5.045
6	1.60	4.096	4.81	2.651	2.164	1.174	3.112	2.541	8.250	5.499	6.736	1.403	4.839	2.998	1.120	4.109	2.375
7	1.83	6.128	4.81	3.133	2.158	.785	2.459	1.694	7.704	3.656	5.307	.618	2.380	1.304	.444	1.777	.930
8	2.33	12.649	4.85	3.615	2.133	.383	1.385	.817	5.007	1.743	2.953	.235	.995	.487	.146	.629	.301
9	3.08	29.218	4.88	4.097	2.092	.167	.684	.349	2.802	.730	1.430	.068	.311	.138	.033	.150	.067
10	4.17	72.512	4.92	4.579	2.035	.068	.311	.138	1.424	.281	.632	0	0	0	0	0	0
					Σ=	6.544 =A	13.174 =B	12.860 =C	34.481 =D	26.859 =E	28.034 =F						

1	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
Point or a	col. 16 × C_2C_4	col. 17 ×C_6	col. 18 ×C_3	col. 19 − col. 20	col. 21 + col. 22	H= col. 23 ÷C_H	col. 24 ×C_6	col. 16 ×C_2	col. 25 + col. 26 − col. 17	V= col. 27 ÷C_3	col. 24 ×C_4	col. 16 ×C_7	col. 29 + col. 30 − col. 17	M_A= col. 31 ÷ 0.1C_8	(col. 28 ×l)− (col. 29 ×h)	l− (10× col. 5)	M_B= −col. 32 +col. 33 −col. 34
0_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	137.487	−73.344	−209.761	210.831	1.070	.137	−.321	+23.371	−8.280	.990	.806	30.845	+.321	−.77	+44.96	+45.79	−.06
2	105.845	−58.628	−161.560	164.473	2.913	.374	−.876	+17.992	−7.928	.948	2.200	23.746	+.902	−2.16	+38.18	+40.97	−.63
3	75.813	−44.144	−115.716	119.957	4.241	.545	−1.276	+12.887	−7.246	.867	3.206	17.009	+1.358	−3.24	+30.84	+36.15	−2.07
4	49.014	−30.435	−74.722	79.449	4.727	.607	−1.421	+8.332	−6.097	.728	3.571	10.996	+1.568	−3.75	+22.89	+31.33	−4.65
5	27.763	−18.590	−42.181	46.353	4.172	.536	−1.255	+4.719	−4.477	.535	3.153	6.229	+1.441	−3.44	+15.02	+26.51	−8.06
6	13.164	−9.619	−19.857	22.783	2.926	.376	−.880	+2.238	−2.751	.329	2.212	2.953	+1.056	−2.52	+8.30	+21.69	−10.87
7	5.219	−4.160	−7.776	9.379	1.603	.206	−.482	+.887	−1.372	.164	1.212	1.171	+.608	−1.45	+3.76	+16.87	−11.66
8	1.716	−1.472	−2.517	3.188	.671	.086	−.201	+.292	−.538	.064	.506	.385	+.262	−.63	+1.36	+12.05	−10.06
9	.388	−.351	−.560	.739	.179	.023	−.054	+.066	−.138	.017	.135	.087	+.072	−.17	+.36	+7.23	−6.70
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+2.41	−2.41

l=48.2 ft. h=20.1 ft. dx=4.82 ft.

$C_1=\frac{C}{B}=+0.984$ $C_6=\frac{CB}{A}-F=-2.341$

$C_2=\frac{B}{A}=+1.998$ $C_7=\frac{D}{B}=+2.637$

$C_3=\frac{B^2}{A}-D=-8.361$ $C_8=\frac{AD}{B}-B=+4.185$

$C_4=\frac{DC}{B}-F=+5.883$ $C_2C_4=+11.754$

$C_5=\frac{FC}{B}-E=+0.716$ $C_H=C_3C_5-C_6C_4=+7.786$

Table 11 (b).—*Fixed-end moments and fixed-end thrusts, span 2*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Point or a	t	(col. 2)^3	ds	$\frac{x}{10}$	$\frac{y}{10}$	col. 4 ÷ col. 3	col. 5× col. 7	col. 6× col. 7	col. 5× col. 8	col. 6× col. 9	col. 5× col. 9	$\sum_{a+1}^{B}$ col. 7	$\sum_{a+1}^{B}$ col. 8	$\sum_{a+1}^{B}$ col. 9	$\sum_{a}^{B}$ col. 13 ×0.1dx	$\sum_{a}^{B}$ col. 14 ×0.1dx	$\sum_{a}^{B}$ col. 15 ×0.1dx
0_1	2.09	9.129	6.00	0	0.300	0.657	0	0.197	0	0.059	0	0	0	0	0	0	0
0_2	2.27	11.697	6.00	0	.900	.513	0	.462	0	.416	0	0	0	0	0	0	0
0_3	2.45	14.706	6.00	0	1.500	.408	0	.612	0	.918	0	0	0	0	0	0	0
0_4	2.63	18.191	6.00	0	2.100	.330	0	.693	0	1.455	0	0	0	0	0	0	0
1′	2.50	15.625	3.33	.160	2.454	.213	.034	.523	.005	1.283	.084	6.001	8.693	15.953	7.734	12.990	20.649
2′	1.96	7.530	3.27	.480	2.540	.434	.208	1.102	.100	2.799	.529	5.567	8.485	14.851	5.813	10.208	15.544
3′	1.62	4.252	3.25	.800	2.604	.764	.611	1.989	.489	5.179	1.591	4.803	7.874	12.862	4.032	7.493	10.792
4′	1.42	2.863	3.23	1.120	2.660	1.128	1.263	3.000	1.415	7.980	3.360	3.675	6.611	9.862	2.495	4.973	6.676
5′	1.35	2.460	3.21	1.440	2.688	1.305	1.879	3.508	2.706	9.430	5.052	2.370	4.732	6.354	1.319	2.858	3.520
6′	1.37	2.571	3.20	1.760	2.694	1.245	2.191	3.354	3.856	9.036	5.903	1.125	2.541	3.000	.561	1.343	1.487
7′	1.67	4.657	3.22	2.080	2.683	.691	1.437	1.854	2.989	4.974	3.856	.434	1.104	1.146	.201	.530	.527
8′	2.25	11.391	3.24	2.400	2.662	.284	.682	.756	1.637	2.012	1.814	.150	.422	.390	.062	.177	.160
9′	3.12	30.371	3.26	2.720	2.618	.107	.291	.280	.792	.733	.762	.043	.131	.110	.014	.042	.035
10′	4.25	76.766	3.29	3.040	2.560	.043	.131	.110	.398	.282	.334	0	0	0	0	0	0
					Σ =	8.122 = A	8.727 = B	18.440 = C	14.387 = D	46.556 = E	23.285 = F						

1	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
Point or a	col. 16× C_2C_4	col. 17× C_6	col. 18× C_3	col. 19− col. 20	col. 21+ col. 22	H=col. 23 ÷ C_H	col. 24× C_5	col. 16× C_2	col. 25+ col. 26− col. 17	V=col. 27 ÷ C_1	col. 24× C_4	col. 16× C_7	col. 29+ col. 30− col. 17	M_A'= col. 31 ÷ 0.1C_8	(col. 28× l)−(col. 29×h)	l−(10× col. 5)	M_B'= −col. 32 −col. 33 +col. 34
0_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1′	59.088	−45.088	−103.451	104.176	.725	.063	−.219	8.306	−4.903	.978	.448	12.753	+.251	+.46	+29.73	+30.40	+.21
2′	44.411	−35.432	−77.875	79.843	1.968	.172	−.597	6.243	−4.562	.911	1.224	9.586	+.602	+1.29	+24.80	+27.20	+1.11
3′	30.804	−26.008	−54.068	56.812	2.744	.240	−.833	4.330	−3.996	.798	1.707	6.649	+.863	+1.85	+19.46	+24.00	+2.69
4′	19.062	−17.261	−33.447	36.323	2.876	.251	−.872	2.680	−3.163	.632	1.786	4.114	+.927	+1.99	+13.87	+20.80	+4.94
5′	10.077	−9.920	−17.635	19.997	2.362	.206	−.716	1.417	−2.157	.431	1.465	2.175	+.786	+1.69	+8.58	+17.60	+7.33
6′	4.286	−4.662	−7.450	8.948	1.498	.131	−.455	.603	−1.195	.239	.932	.925	+.514	+1.10	+4.33	+14.40	+8.97
7′	1.536	−1.840	−2.640	3.376	.736	.064	−.222	.216	−.536	.107	.455	.331	+.256	+.55	+1.80	+11.20	+8.85
8′	.474	−.614	−.802	1.089	.287	.025	−.087	.067	−.197	.039	.178	.102	+.103	+.22	+.62	+8.00	+7.16
9′	.107	−.146	−.175	.253	.078	.007	−.024	.015	−.051	.010	.050	.023	+.031	+.07	+.15	+4.80	+4.58
10′	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+1.60	+1.60

l=32.00 ft. h=25.30 ft dx=3.20 ft.

$$C_1=\frac{C}{B}=+2.113 \qquad C_6=\frac{CB}{A}-F=-3.471$$

$$C_2=\frac{B}{A}=+1.074 \qquad C_7=\frac{D}{B}=+1.649$$

$$C_3=\frac{B^2}{A}-D=-5.010 \qquad C_8=\frac{AD}{B}-B=+4.663$$

$$C_4=\frac{DC}{B}-F=+7.114 \qquad C_2C_4=+7.640$$

$$C_5=\frac{FC}{B}-E=+2.645 \qquad C_H=C_3C_5-C_6C_4=+11.442$$

Table 11 (c).—*Calculation of moments, vertical reactions, and thrusts*

Expression	Value, span 1	Value, span 2
$H_1=\frac{(lC_1-h)100C_3-(10C_2-l)100C_4}{-10{,}000C_H}=$	0.0802	0.0531
$V_1=\frac{H_1(100C_6)+10C_2-l}{100C_3}=$	.0562	.0793
$M_{1A}=\frac{H_1(100C_4)+10C_7-l}{10C_8}=$	.606	.478
$M_{1B}=M_{1A}+V_1l-H_1h=$	1.703	1.669
$H_2=\frac{h(100C_3)}{10{,}000C_H}=$	.216	.111
$V_2=H_2\left(\frac{C_6}{C_3}\right)=$	.0605	.0769
$M_{2A}=H_2\left(\frac{100C_4}{10C_8}\right)=$	3.036	1.693
$M_{2B}=M_{2A}+V_2l-H_2h=$	1.610	1.346

Table 11 (d).—*Calculation of arch rib joint constants*

Expression	Value, span 1	Value, span 2
Moment stiffness $=M_{1B}\div 12=$	0.142	0.139
Induced thrust $=H_1\div M_{1B}=$	.047	.032
Thrust stiffness $=H_2\div 12h=$	.00090	.00037
Induced moment (at pier top) $=M_{2B}\div H_2=$	7.45	12.13
Induced moment (at footing) $=-M_{2A}\div H_2=$	−14.06	−15.25
Moment carry-over $=-M_{1A}\div M_{1B}=$	−.35	−.29

Table 12.—*Pier constants*

$l=22.5$ ft. $t=2.0$ ft.

$I=t^3/12=0.667$

Expression	Value
Moment stiffness $=4I\div l=$	0.119
Induced thrust $=3\div 2l=$	.067
Thrust stiffness $=12I\div l^3=$	.00070
Induced moment $=l\div 2=$	11.25
Moment carry-over $=$	+.50

Table 13.—*Distribution of moment and thrust stiffness*

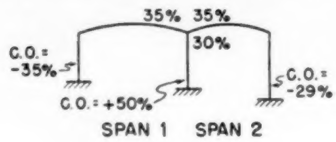
Member	Moment stiffness		Thrust stiffness	
	Value	Distribution factor	Value	Distribution factor
		Percent		*Percent*
Arch rib, span 1	0.142	35	0.00090	45
Arch rib, span 2	.139	35	.00037	19
Pier	.119	30	.00070	36
Total	.400	100	.00197	100

Table 14.—*Distribution of induced moments, induced thrusts, and moment carry-overs*

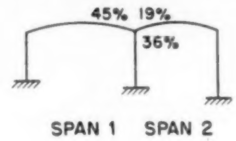
Member	Induced moment— At pier top (multiply value by distributed thrusts)	Induced moment— At footing (multiply value by distributed thrusts)	Induced thrust (multiply value by distributed moments)	Moment carry-over
				Percent
Arch rib, span 1	+7.45	−14.06	+0.046	−35
Arch rib, span 2	+12.13	−15.25	+.032	−29
Pier	−11.25	+11.25	−.067	+50

Tables 15 & 16.—*Unit moment and thrust distributions, unit loads in span 1*

MOMENT DISTRIBUTION FACTORS



THRUST DISTRIBUTION FACTORS



INDUCED THRUSTS

0.046×moment (arch rib, span 1)
0.032×moment (arch rib, span 2)
−0.067×moment (pier)

INDUCED MOMENTS

7.44 (−14.06 at footing)×thrust (arch rib, span 1)
12.13 (−15.25 at footing)×thrust (arch rib, span 2)
−11.25 (−11.25 at footing)×thrust (pier)

Table 15.—*Distribution of unit moment, span 1*

	Rib	Pier	Rib		Rib	Pier	Rib	
Fixed-end moment....	−1.000	0	0		0	0	0	...Fixed-end thrust
Balancing moment....	+.350	+.300	+.350	Induced thrust →	+.016	−.020	+.011	
	−.022	+.034	−.012	←Induced moment	−.003	−.003	−.001	...Balancing thrust
Balancing moment....	......	......	......	Induced thrust→	......	......	......	
	......	......	......	←Induced moment	......	......	......	...Balancing thrust
Final moments........	−.672	+.334	+.338		+.013	−.023	+.010	......Final thrusts

	Rib	Pier	Rib	Rib	Pier	Rib	
Sum of balancing thrusts......	−0.003	−0.003	−0.001	+0.042	+0.034	+0.015	....Moment induced at footing
Sum of balancing moments....	+.350	+.300	+.350	−.123	+.150	−.102	....Moment carried over to footing
	Tota moment at footing..			−.081	+.184	−.087	

Footing moments

Table 16.—*Distribution of unit thrust, span 1*

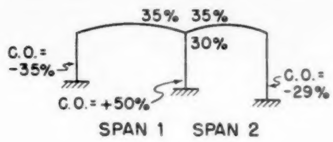
	Rib	Pier	Rib		Rib	Pier	Rib	
Fixed-end moment....	0	0	0		+1.000	0	0	...Fixed-end thrust
	−3.348	+4.050	−2.305	←Induced moment	−.450	−.360	−.190	...Balancing thrust
Balancing moment....	+.561	+.481	+.561	Induced thrust→	+.026	−.032	+.018	
	−.037	+.045	−.036	←Induced moment	−.005	−.004	−.003	...Balancing thrust
Balancing moment....	+.010	+.008	+.010	Induced thrust→	......	......	......	
Final moments........	−2.814	+4.584	−1.770		+.571	−.396	−.175	........Final thrusts

	Rib	Pier	Rib	Rib	Pier	Rib	
Sum of balancing thrusts......	−0.455	−0.364	−0.193	+6.397	+4.095	+2.943	...Moment induced at footing
Sum of balancing moments....	+.571	+.489	+.571	−.200	+.245	−.166	...Moment carried over to footing
	Total moment at footing...			+6.197	+4.340	+2.777	

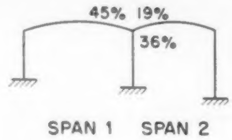
Footing moments

Tables 17 & 18.—*Unit moment and thrust distributions, unit loads in span 2*

MOMENT DISTRIBUTION FACTORS



THRUST DISTRIBUTION FACTORS



INDUCED THRUSTS

+0.046×moment (arch rib, span 1)
+0.032×moment (arch rib, span 2)
−0.067×moment (pier)

INDUCED MOMENTS

+7.44 (−14.06 at footing)×thrust (arch rib, span 1)
+12.13 (−15.25 at footing)×thrust (arch rib, span 2)
−11.25 (−11.25 at footing)×thrust (pier)

Table 17.—*Distribution of unit moment, span 2*

	Rib	Pier	Rib		Rib	Pier	Rib	
Fixed-end moment	0	0	+1.000		0	0	0	Fixed-end thrust
Balancing moment	−.350	−.300	−.350	Induced thrust→	−.016	+.020	−.011	
	+.022	−.034	+.012	←Induced moment	+.003	+.003	+.001	Balancing thrust
Balancing moment	------	------	------	Induced thrust→	------	------	------	
	------	------	------	←Induced moment	------	------	------	Balancing thrust
Final moments	−.328	−.334	+.662		−.013	+.023	−.010	Final thrusts

	Rib	Pier	Rib	Rib	Pier	Rib	
Sum of balancing thrusts	+0.003	+0.003	+0.001	−0.042	−0.034	−0.015	Moment induced at footing
Sum of balancing moments	−.350	−.300	−.350	+.123	−.150	+.102	Moment carried over to footing
	Total moment at footing			+.081	−.184	+.087	

Footing moments

Table 18.—*Distribution of unit thrust, span 2*

	Rib	Pier	Rib		Rib	Pier	Rib	
Fixed-end moment	0	0	0		0	0	−1.000	Fixed-end thrust
	+3.348	−4.050	+2.305	←Induced moment	+.450	+.360	+.190	Balancing thrust
Balancing moment	−.561	−.481	−.561	Induced thrust→	−.026	+.032	−.018	
	+.037	−.045	+.036	←Induced moment	+.005	+.004	+.003	Balancing thrust
Balancing moment	−.010	−.008	−.010	Induced thrust→	------	------	------	
inal moments	+2.814	−4.584	+1.770		+.429	+.396	−.825	Final thrusts

	Rib	Pier	Rib	Rib	Pier	Rib	
Sum of balancing thrusts	+0.455	+0.364	+0.193	−6.397	−4.095	−2.943	Moment induced at footing
Sum of balancing moments	−.571	−.489	−.571	+.200	−.245	+.166	Moment carried over to footing
	Total moment at footing			−6.197	−4.340	−2.777	

Footing moments

Table 19.—*Tabulation of final moments and thrusts, span 1*

Table 19 (a)

Fixed-end moment = −1.000				
M_B	$M_{B'}$	H_1	H_2	H_3
−0.672	+0.338	+0.013	−0.023	+0.010

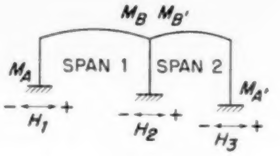


Table 19 (b)

Fixed-end thrust = +1.000				
M_B	$M_{B'}$	H_1	H_2	H_3
−2.814	−1.770	+0.571	−0.396	−0.175

Table 19 (c)

Unit load at point—	Fixed-end moment	Fixed-end thrust	M and H due to fixed-end moment					M and H due to fixed-end thrust					Final values of M and H				
			M_B	$M_{B'}$	H_1	H_2	H_3	M_B	$M_{B'}$	H_1	H_2	H_3	M_B	$M_{B'}$	H_1	H_2	H_3
1	−0.06	+0.137	−0.04	+0.02	+0.001	−0.001	+0.001	−0.39	−0.24	+0.078	−0.054	−0.024	−0.43	−0.22	+0.079	−0.055	−0.023
2	−.63	+.374	−.42	+.21	+.008	−.014	+.006	−1.05	−.66	+.214	−.148	−.065	−1.47	−.45	+.222	−.162	−.059
3	−2.07	+.545	−1.39	+.70	+.027	−.048	+.021	−1.53	−.96	+.311	−.215	−.095	−2.92	−.26	+.338	−.263	−.074
4	−4.69	+.507	−3.15	+1.59	+.061	−.108	+.047	−1.71	−1.07	+.347	−.240	−.106	−4.86	+.52	+.408	−.348	−.059
5	−8.06	+.536	−5.42	+2.72	+.105	−.185	+.081	−1.51	−.95	+.306	−.212	−.094	−6.93	+1.77	+.411	−.397	−.013
6	−10.87	+.376	−7.30	+3.67	+.141	−.250	+.109	−1.06	−.67	+.215	−.149	−.066	−8.36	+3.00	+.356	−.399	+.043
7	−11.66	+.206	−7.84	+3.94	+.152	−.268	+.117	−.58	−.36	+.118	−.082	−.036	−8.42	+3.58	+.270	−.350	+.081
8	−10.06	+.086	−6.76	+3.40	+.131	−.231	+.101	−.24	−.15	+.049	−.034	−.015	−7.00	+3.25	+.180	−.265	+.086
9	−6.70	+.023	−4.50	+2.26	+.087	−.154	+.067	−.06	−.04	+.013	−.009	−.004	−4.56	+2.22	+.100	−.163	+.063
10	−2.41	0	−1.62	+.81	+.031	−.055	+.024	0	0	0	0	0	−1.62	+.81	+.031	−.055	+.024

Table 20.—*Tabulation of final moments and thrusts, span 2*

Table 20 (a)

Fixed-end moment = +1.000				
M_B	$M_{B'}$	H_1	H_2	H_3
−0.328	+0.662	−0.013	+0.023	−0.010

Table 20 (b)

Fixed-end thrust = −1.000				
M_B	$M_{B'}$	H_1	H_2	H_3
+2.814	+1.770	+0.429	+0.396	−0.825

Table 20 (c)

Unit load at point—	Fixed-end moment	Fixed-end thrust [1]	M and H due to fixed-end moment					M and H due to fixed-end thrust					Final values of M and H				
			M_B	$M_{B'}$	H_1	H_2	H_3	M_B	$M_{B'}$	H_1	H_2	H_3	M_B	$M_{B'}$	H_1	H_2	H_3
10′	+1.60	0	−0.52	+1.06	−0.021	+0.037	−0.016	0	0	0	0	0	−0.52	+1.06	−0.021	+0.037	−0.016
9′	+4.58	−.007	−1.52	+3.03	−.060	+.105	−.046	+.02	+.01	+.003	+.003	−.006	−1.48	+3.04	−.057	+.108	−.052
8′	+7.16	−.025	−2.35	+4.74	−.093	+.165	−.072	+.07	+.04	+.011	+.010	−.021	−2.28	+4.78	−.082	+.175	−.093
7′	+8.85	−.064	−2.90	+5.86	−.115	+.204	−.089	+.18	+.11	+.027	+.025	−.053	−2.72	+5.97	−.088	+.229	−.142
6′	+8.97	−.131	−2.94	+5.94	−.117	+.206	−.090	+.37	+.23	+.056	+.052	−.108	−2.57	+6.17	−.061	+.758	−.198
5′	+7.33	−.206	−2.40	+4.85	−.095	+.169	−.073	+.58	+.36	+.088	+.082	−.170	−1.82	+5.21	−.007	+.251	−.243
4′	+4.94	−.251	−1.62	+3.27	−.164	+.114	−.049	+.71	+.44	+.108	+.099	−.207	−.91	+3.71	+.044	+.213	−.256
3′	+2.69	−.240	−.88	+1.78	−.035	+.062	−.027	+.68	+.42	+.103	+.095	−.198	−.20	+2.20	+.068	+.157	−.225
2′	+1.11	−.172	−.36	+.73	−.014	+.026	−.011	+.48	+.30	+.074	+.068	−.142	+.12	+1.03	+.060	+.094	−.153
1′	+.21	−.163	−.07	+.14	−.003	+.005	−.002	+.18	+.11	+.027	+.025	−.052	+.11	+.25	+.024	+.030	−.054

[1] Negative sign used to conform to sign convention.

Table 21.—*Tabulation of final moments at footings, span 1*

Table 21 (a)

Fixed-end moment = −1.000		
M_1	M_2	M_3
−0.081	+0.184	−0.087

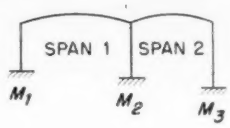


Table 21 (b)

Fixed-end thrust = +1.000		
M_1	M_2	M_3
+6.197	+4.340	+2.777

Table 21 (c)

Unit load at point	Fixed-end moment (M_B)	Fixed-end thrust	M_A	M due to fixed-end moment			M due to fixed-end thrust			Final values of M		
				M_1	M_2	M_3	M_1	M_2	M_3	M_1+M_A	M_2	M_3
1	−0.06	+0.137	−0.77	−0.005	+0.01	−0.005	+0.85	+0.59	+0.38	+0.07	+0.60	+0.37
2	−.63	+.374	−2.16	−.05	+.12	−.06	+2.32	+1.62	+1.04	+.11	+1.74	+.98
3	−2.07	+.545	−3.24	−.17	+.38	−.18	+3.38	+2.37	+1.51	−.03	+2.75	+1.33
4	−4.69	+.607	−3.75	−.38	+.86	−.41	+3.76	+2.63	+1.69	−.37	+3.49	+1.28
5	−8.06	+.536	−3.44	−.65	+1.48	−.70	+3.32	+2.33	+1.49	−.77	+3.81	+.79
6	−10.87	+.376	−2.52	−.88	+2.00	−.95	+2.33	+1.63	+1.04	−1.07	+3.63	+.09
7	−11.66	+.206	−1.45	−.94	+2.15	−1.01	+1.28	+.89	+.57	−1.11	+3.04	−.44
8	−10.06	+.086	−.63	−.81	+1.85	−.88	+.53	+.37	+.24	−.91	+2.22	−.64
9	−6.70	+.023	−.17	−.54	+1.23	−.58	+.14	+.10	+.06	−.57	+1.33	−.52
10	−2.41	0	0	−.20	+.44	−.21	0	0	0	−.20	+.44	−.21

Table 22.—*Tabulation of final moments at footings, span 2*

Table 22 (a)

Fixed-end moment = +1.000		
M_1	M_2	M_3
+0.081	−0.184	+0.087

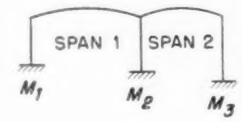


Table 22 (b)

Fixed-end thrust = −1.000		
M_1	M_2	M_3
−6.197	−4.340	−2.777

Table 22 (c)

Unit load at point	Fixed-end moment (M_B')	Fixed-end thrust	M_A'	M due to fixed-end moment			M due to fixed-end thrust			Final values of M		
				M_1	M_2	M_3	M_1	M_2	M_3	M_1	M_2	M_3+M_A'
10′	+1.60	0	0	+0.13	−0.29	+0.14	0	0	0	+0.13	−0.29	+0.14
9′	+4 58	−.007	+.07	+.37	−.84	+.40	−.04	−.03	−.02	+.33	−.87	+.45
8′	+7.16	−.025	+.22	+.58	−1.32	+.62	−.15	−.11	−.07	+.43	−1.43	+.77
7′	+8.85	−.064	+.55	+.72	−1.63	+.77	−.40	−.28	−.18	+.32	−1.91	+1.14
6′	+8.97	−.131	+1.10	+.73	−1.65	+.78	−.81	−.57	−.36	−.08	−2.22	+1.52
5′	+7.33	−.206	+1.69	+.59	−1.35	+.64	−1.28	−.89	−.57	−.69	−2.24	+1.76
4′	+4.94	−.251	+1.99	+.40	−.91	+.43	−1.56	−1.09	−.70	−1.16	−2.00	+1.72
3′	+2.69	−.240	+1.85	+.22	−.49	+.23	−1.49	−1.04	−.67	−1.27	−1.53	+1.41
2′	+1.11	−.172	+1.29	+.09	−.20	+.10	−1.07	−.75	−.48	−.98	−.95	+.91
1′	+.21	−.063	+.46	+.02	−.04	+.02	−.40	−.27	−.17	−.38	−.31	+.31

New Publications

THE IDENTIFICATION OF ROCK TYPES

To meet popular demand a convenient 6 x 9-inch reprint has been made of the article *The Identification of Rock Types,* by D. O. Woolf, which appeared in Public Roads, vol. 26, No. 2, June 1950. The article presents a simple method for use by the highway engineer in making field identification of the different types of rock with which he is concerned. It will be extremely useful to engineers, engineering students, and others whose work requires a limited, practical knowledge of geology. The reprint is for sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C., at 10 cents a copy.

A BIBLIOGRAPHY OF HIGHWAY PLANNING REPORTS

The Bureau of Public Roads recently published a 48-page *Bibliography of Highway Planning Reports,* which is now for sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C., at 30 cents a copy. The bibliography covers the period 1930 to date, and includes listings of Nation-wide, State, and city highway planning reports such as those of State-wide highway planning surveys and of traffic, origin-destination, design, and highway needs studies. The reports range from long-term studies of State-wide scope to discussions and plans for individual routes, and are the work of the Bureau of Public Roads and State, city, and consulting engineers.

The interest in highway planning continues to increase. This bibliography makes available a listing of reports on the subject, useful both to those interested in the general field of planning and to those concerned with a particular State, city, or route.

COMPARATIVE EFFECT OF HINGED AND FIXED FOOTINGS AT CRITICAL POINTS

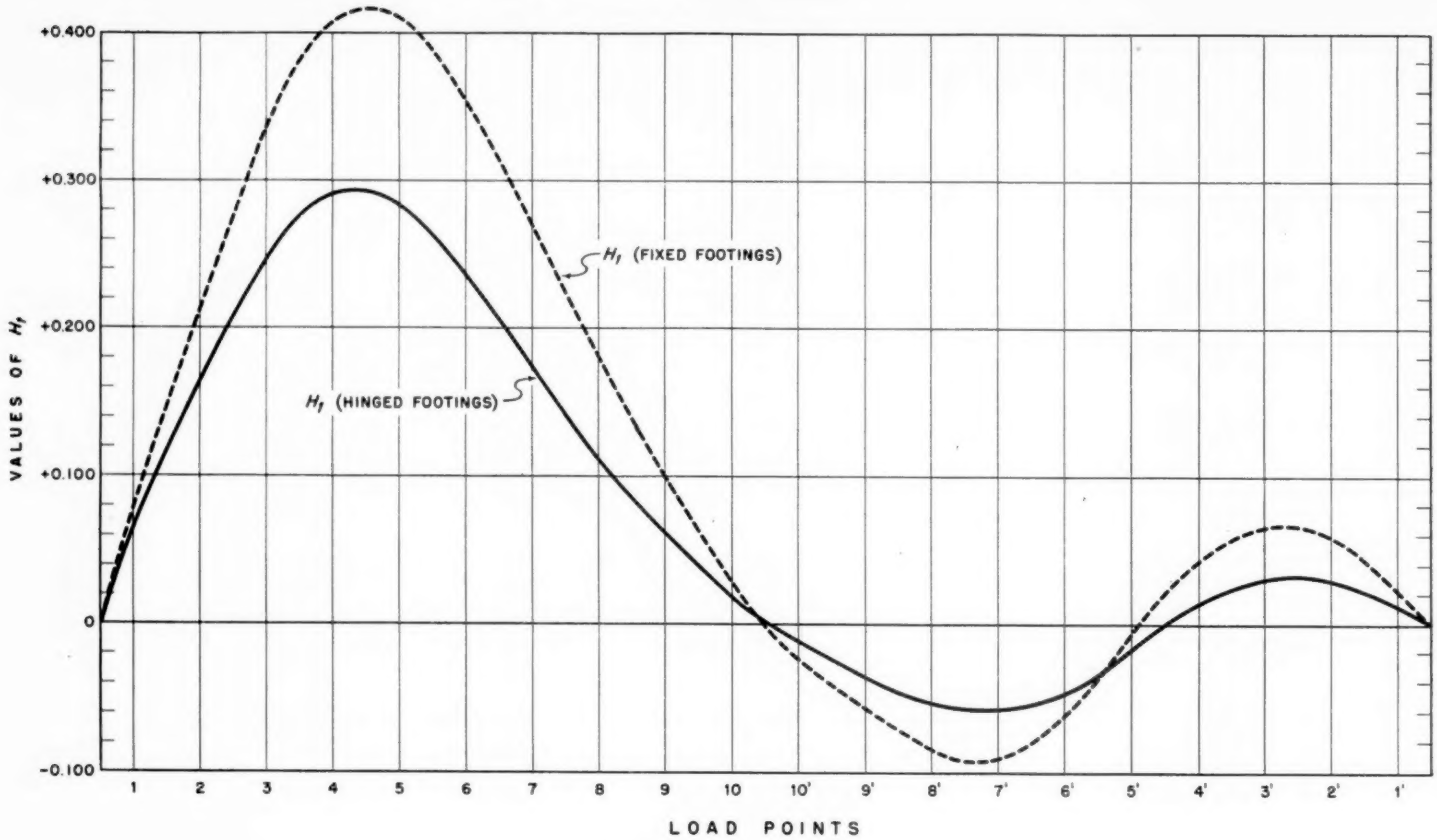


Figure 13.—Influence line for H_1.

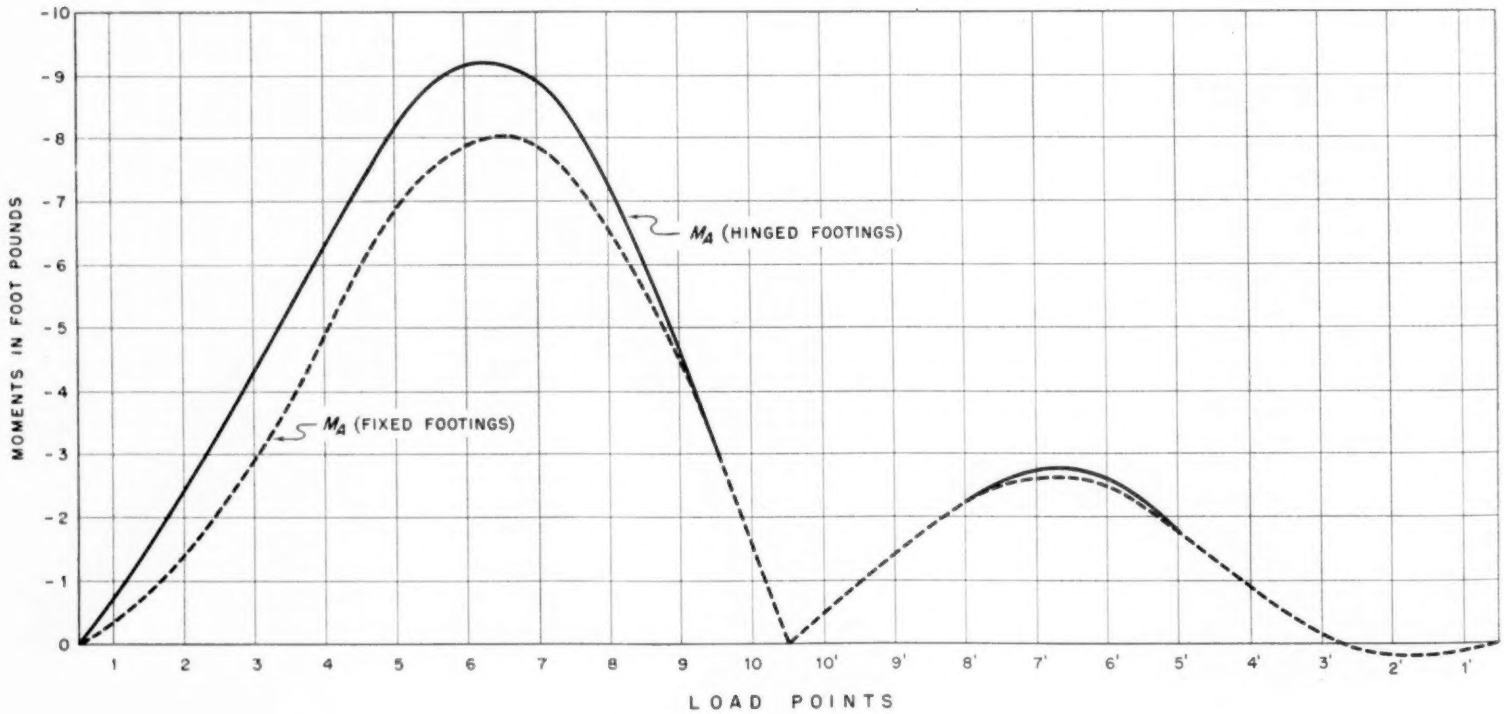


Figure 14.—Influence line for M_A.

U. S. GOVERNMENT PRINTING OFFICE: 1950

A complete list of the publications of the Bureau of Public Roads, classified according to subject and including the more important articles in PUBLIC ROADS, may be obtained upon request addressed to Bureau of Public Roads, Washington 25, D. C.

PUBLICATIONS of the Bureau of Public Roads

The following publications are sold by the Superintendent of Documents, Government Printing Office, Washington 25, D. C. Orders should be sent direct to the Superintendent of Documents. Prepayment is required.

ANNUAL REPORTS

(See also adjacent column)

Reports of the Chief of the Bureau of Public Roads:
1937, 10 cents. 1938, 10 cents. 1939, 10 cents.

Work of the Public Roads Administration:
1940, 10 cents. 1942, 10 cents. 1948, 20 cents.
1941, 15 cents. 1946, 20 cents. 1949, 25 cents.
1947, 20 cents.

HOUSE DOCUMENT NO. 462

Part 1 . . . Nonuniformity of State Motor-Vehicle Traffic Laws. 15 cents.

Part 2 . . . Skilled Investigation at the Scene of the Accident Needed to Develop Causes. 10 cents.

Part 3 . . . Inadequacy of State Motor-Vehicle Accident Reporting. 10 cents.

Part 4 . . . Official Inspection of Vehicles. 10 cents.

Part 5 . . . Case Histories of Fatal Highway Accidents. 10 cents.

Part 6 . . . The Accident-Prone Driver. 10 cents.

UNIFORM VEHICLE CODE

Act I.—Uniform Motor-Vehicle Administration, Registration, Certificate of Title, and Antitheft Act. 10 cents.

Act II.—Uniform Motor-Vehicle Operators' and Chauffeurs' License Act. 10 cents.

Act III.—Uniform Motor-Vehicle Civil Liability Act. 10 cents.

Act IV.—Uniform Motor-Vehicle Safety Responsibility Act. 10 cents.

Act V.—Uniform Act Regulating Traffic on Highways. 20 cents.

Model Traffic Ordinance. 15 cents.

MISCELLANEOUS PUBLICATIONS

Bibliography of Highway Planning Reports. 30 cents.
Construction of Private Driveways (No. 272MP). 10 cents.
Economic and Statistical Analysis of Highway Construction Expenditures. 15 cents.
Electrical Equipment on Movable Bridges (No. 265T). 40 cents.
Federal Legislation and Regulations Relating to Highway Construction. 40 cents.
Financing of Highways by Counties and Local Rural Governments, 1931–41. 45 cents.
Guides to Traffic Safety. 10 cents.
Highway Accidents. 10 cents.
Highway Bridge Location (No. 1486D). 15 cents.
Highway Capacity Manual. 65 cents.
Highway Needs of the National Defense (House Document No. 249). 50 cents.
Highway Practice in the United States of America. 50 cents.
Highway Statistics, 1945. 35 cents.
Highway Statistics, 1946. 50 cents.
Highway Statistics, 1947. 45 cents.
Highway Statistics, 1948. 65 cents.
Highway Statistics, Summary to 1945. 40 cents.
Highways of History. 25 cents.
Identification of Rock Types. 10 cents.
Interregional Highways (House Document No. 379). 75 cents.
Legal Aspects of Controlling Highway Access. 15 cents.
Manual on Uniform Traffic Control Devices for Streets and Highways. 50 cents.
Principles of Highway Construction as Applied to Airports, Flight Strips, and Other Landing Areas for Aircraft. $1.50.
Public Control of Highway Access and Roadside Development. 35 cents.
Public Land Acquisition for Highway Purposes. 10 cents.
Roadside Improvement (No. 191MP). 10 cents.
Specifications for Construction of Roads and Bridges in National Forests and National Parks (FP-41). $1.25.
Taxation of Motor Vehicles in 1932. 35 cents.
The Local Rural Road Problem. 20 cents.
Tire Wear and Tire Failures on Various Road Surfaces. 10 cents.
Transition Curves for Highways. $1.25.

Single copies of the following publications are available to highway engineers and administrators for official use, and may be obtained by those so qualified upon request addressed to the Bureau of Public Roads. They are not sold by the Superintendent of Documents.

ANNUAL REPORTS

(See also adjacent column)

Public Roads Administration Annual Reports:
1943. 1944. 1945.

MISCELLANEOUS PUBLICATIONS

Bibliography on Automobile Parking in the United States.
Bibliography on Highway Lighting.
Bibliography on Highway Safety.
Bibliography on Land Acquisition for Public Roads.
Bibliography on Roadside Control.
Express Highways in the United States: a Bibliography.
Indexes to PUBLIC ROADS, volumes 17–19, 22, and 23.
Road Work on Farm Outlets Needs Skill and Right Equipment.

STATUS OF FEDERAL-AID HIGHWAY PROGRAM

AS OF AUGUST 31, 1950

(Thousand Dollars)

STATE	UNPROGRAMMED BALANCES	ACTIVE PROGRAM											
		PROGRAMMED ONLY			PLANS APPROVED, CONSTRUCTION NOT STARTED			CONSTRUCTION UNDER WAY			TOTAL		
		Total Cost	Federal Funds	Miles	Total Cost	Federal Funds	Miles	Total Cost	Federal Funds	Miles	Total Cost	Federal Funds	Miles
Alabama	$13,306	$12,611	$6,293	389.7	$4,145	$1,971	101.8	$13,069	$6,729	319.6	$29,825	$14,993	811.1
Arizona	724	3,447	2,410	75.8	1,094	732	20.8	6,246	4,478	118.0	10,787	7,620	214.6
Arkansas	1,775	8,901	5,109	213.2	6,390	3,101	190.7	17,192	8,472	435.1	32,483	16,682	839.0
California	3,115	30,914	12,388	194.2	4,735	2,395	57.0	39,106	19,244	247.5	74,755	34,027	498.7
Colorado	2,728	3,931	2,115	61.2	2,580	1,459	83.3	15,227	8,777	295.4	21,738	12,351	439.9
Connecticut	2,162	9,059	4,312	24.6	2,011	1,217	3.2	7,236	4,106	12.4	18,306	9,635	40.2
Delaware	1,450	1,676	847	22.4	1,391	694	18.5	5,567	2,669	54.6	8,634	4,210	95.5
Florida	4,068	15,795	7,962	441.3	6,150	3,248	162.9	11,566	5,825	273.4	33,511	17,035	877.6
Georgia	1,390	18,799	9,642	500.2	7,875	3,934	191.4	35,937	16,926	740.7	62,611	30,502	1,432.3
Idaho	4,221	8,499	5,353	293.1	2,698	1,008	90.5	6,513	4,134	189.6	17,710	10,495	573.2
Illinois	20,302	42,514	22,436	369.4	11,218	5,614	110.6	53,161	25,548	370.1	106,893	53,598	850.1
Indiana	13,818	20,456	10,498	101.3	6,576	3,456	44.8	14,993	7,784	82.3	42,025	21,738	228.4
Iowa	3,280	11,016	4,072	392.9	4,910	1,771	232.4	20,916	10,258	790.2	36,842	16,101	1,415.5
Kansas	4,376	6,492	3,115	907.6	7,492	3,771	561.3	12,907	6,597	546.7	26,891	13,483	2,015.6
Kentucky	1,796	13,191	6,427	143.6	6,510	3,214	140.5	18,588	9,159	336.1	38,289	18,800	620.2
Louisiana	4,318	20,257	8,942	177.8	8,634	4,213	109.6	17,810	9,267	203.6	46,701	22,422	491.0
Maine	1,692	7,964	4,194	105.4	807	514	5.5	6,965	3,557	81.7	15,736	8,265	192.6
Maryland	1,620	7,897	3,698	29.2	2,206	921	19.5	17,028	8,243	61.9	27,131	12,862	110.6
Massachusetts	2,651	7,041	2,721	6.6	9,703	4,925	9.8	61,257	29,918	62.7	78,001	37,564	79.1
Michigan	4,788	15,289	7,817	437.0	9,613	4,998	301.5	43,320	17,991	369.5	68,222	30,806	1,108.0
Minnesota	1,779	9,877	5,623	880.5	4,416	2,216	383.4	23,834	12,717	883.3	38,127	20,556	2,147.2
Mississippi	5,416	15,499	7,817	621.1	1,483	743	61.8	7,351	3,755	241.9	24,333	12,315	924.8
Missouri	6,767	29,156	15,968	754.2	5,965	2,576	129.4	32,343	16,053	620.7	67,464	34,597	1,504.3
Montana	5,199	14,876	7,700	493.2	2,800	1,661	77.4	15,064	9,386	482.5	32,740	18,747	1,053.1
Nebraska	4,312	17,693	9,249	601.5	5,521	2,512	105.3	12,852	7,353	305.0	36,066	19,114	1,011.8
Nevada	1,328	4,238	3,493	139.0	1,262	1,043	42.1	5,048	4,123	192.3	10,548	8,659	373.4
New Hampshire	1,715	5,502	2,686	46.3	608	298	7.3	3,010	1,455	23.8	9,120	4,439	77.4
New Jersey	3,020	4,195	1,985	7.9	1,058	529	3.8	20,218	9,643	27.7	25,471	12,157	39.4
New Mexico	1,539	7,242	4,620	247.3	3,053	1,967	79.7	6,320	4,131	214.1	16,615	10,718	541.1
New York	27,498	78,306	38,466	225.8	21,959	8,936	32.3	93,182	45,114	187.5	193,447	92,516	445.6
North Carolina	1,516	21,608	10,395	537.9	5,105	2,437	170.0	22,094	10,526	558.3	48,807	23,358	1,266.2
North Dakota	3,080	8,473	4,385	1,242.7	2,816	1,413	250.8	8,903	4,436	599.1	20,192	10,234	2,092.6
Ohio	9,485	41,567	20,114	376.9	14,753	6,753	133.6	54,043	26,873	256.0	110,363	53,740	766.5
Oklahoma	1,176	14,688	8,332	173.0	13,682	6,650	303.0	17,332	8,355	423.7	45,702	23,337	899.7
Oregon	787	5,258	3,022	64.2	2,710	1,530	37.0	11,939	6,783	184.5	19,907	11,335	285.7
Pennsylvania	5,364	23,620	11,721	63.4	19,413	9,680	43.9	82,147	40,620	196.1	125,180	62,021	303.4
Rhode Island	184	6,848	3,422	54.7	4,023	2,073	6.3	9,826	4,896	5.3	20,697	10,391	66.3
South Carolina	3,224	8,202	3,995	209.3	1,830	1,008	95.3	9,749	4,948	374.9	19,781	9,951	679.5
South Dakota	1,229	8,848	5,051	859.5	3,944	2,456	316.5	8,768	5,321	751.7	21,560	12,828	1,927.7
Tennessee	1,087	14,818	7,015	308.0	8,195	3,948	196.2	18,105	8,239	362.5	41,118	19,202	866.7
Texas	5,398	4,756	2,493	296.6	13,689	7,313	328.9	52,761	24,471	1,516.2	71,206	34,277	2,141.7
Utah	1,534	4,411	3,233	125.3	1,111	799	56.9	6,591	4,824	173.5	12,113	8,856	355.7
Vermont	998	2,577	1,412	49.1	962	479	13.6	4,640	2,266	36.9	8,179	4,157	99.6
Virginia	4,170	24,908	12,489	579.5	5,146	2,553	173.5	15,399	7,560	259.5	45,453	22,602	1,012.5
Washington	530	14,810	6,601	92.1	2,305	1,055	75.6	20,448	9,386	157.5	37,563	17,042	325.2
West Virginia	1,857	16,248	6,777	161.5	1,649	920	27.0	9,812	4,914	82.5	27,709	12,611	271.0
Wisconsin	8,825	18,217	9,529	298.2	4,383	2,176	208.0	17,629	8,476	426.4	40,229	20,181	932.6
Wyoming	845	2,046	1,353	34.2	1,602	837	34.7	7,310	4,801	263.0	10,958	6,991	331.9
Hawaii	1,114	8,511	3,809	21.8	2,258	998	12.5	2,941	1,454	14.2	13,710	6,261	48.5
District of Columbia	1,399	5,178	2,951	6.0	340	170	.9	647	324	.9	6,165	3,445	7.8
Puerto Rico	856	14,505	6,563	71.2	728	324	2.7	10,024	4,220	37.7	25,257	11,107	111.6
TOTAL	206,811	722,430	362,620	14,528.4	265,507	131,209	5,865.0	1,034,934	517,105	15,450.3	2,022,871	1,010,934	35,843.7

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